Supersymmetric probes of wrapped M5-brane backgrounds

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A Thesis presented for the degree of
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Dedicated to

my mum,

Ruth, Juan and Lucia,

and the memory of my late father.

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Abstract

In this thesis we consider supersymmetric probes in backgrounds sourced by an M5-brane which is wrapped on holomorphic 2-cycles in $\mathbf{C}^2$ and $\mathbf{C}^3$, respectively. For the first case, we use M2-brane probes to compute the BPS spectra of the corresponding $\mathcal{N} = 2$ gauge theory, as well as M5-brane probes to calculate field theory parameters such as the gauge coupling, theta angle and complex scalar moduli space metric. This background describes a large class of Hanany-Witten models when dimensionally reduced to Type IIA ten-dimensional supergravity. We calculate the instanton action using a Euclidean D0-brane probe in this limit. For the case of an M5-brane wrapping a 2-cycle in $\mathbf{C}^3$, we firstly show an alternative method of deriving this solution which involves the projection conditions and certain spinor bilinear differential equations. We also consider M5-brane probes in this background, and analyse the corresponding $\mathcal{N} = 1$ MQCD gauge theory parameters, in direct analogy with the $\mathcal{N} = 2$ case. We then move on to consider the central charges of the supersymmetry algebra of brane probes in the two backgrounds under consideration. For the case of an M5-brane wrapping a 2-cycle in $\mathbf{C}^2$, we find it allows for M2-branes representing BPS monopoles and vortices. There is also the possibility of a "hidden" M5-brane which is similar to the M2-brane, but which includes a rotation in the complex structure and an extra volume modulus. For the $\mathcal{N} = 1$ case, we find it allows for a supersymmetric M5-brane probe wrapping a Cayley calibrated 4-cycle, which is interpreted as a system of intersecting domain walls. These results are geometrically linked to M-theory structure groups.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

Chapters 1 and 2 of this thesis contain a review of the necessary background material. Chapter 3 is based on joint work with my supervisor Dr. Douglas J. Smith [1], which has been submitted to JHEP. Chapter 4 is a review of background material. Chapter 5 is original work done jointly with my supervisor [2], to be submitted shortly.

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Chapter 1

Introduction

The search for an understanding of the underlying principles that govern the behaviour of our Universe has preoccupied numerous minds over many years. It is a fascinating and ambitious adventure which has seen great breakthroughs despite its inherent difficulty. As thoughts on these matters have become more refined, and experimental evidence more precise, an ever increasing range of physical phenomena is now thought to be understood.

In this struggle to understand the physical, the language of mathematics has been an invaluable ally. That mathematics turns out to be the language of Nature to such an extraordinary degree is a profound mystery. Nevertheless, it has been a recurring theme in this adventure that advances in mathematics have yielded a sharper and more insightful glimpse into the deep physical principles of our Universe.

At present, there are a few theories which attempt to describe the known forces of Nature in a unified manner. Foremost amongst them, at least in popularity, is what is commonly referred to as string theory. Essentially, string theory proposes that the fundamental constituents of Nature are tiny vibrating strings rather than the traditional point particle view of the Universe. Although many of its properties are still obscure, it is thought to provide a comprehensive description of quantum mechanical as well as gravitational phenomena. Despite still having some unresolved fundamental issues, the theory possesses an extraordinary degree of mathematical harmony. Like no other theory before it, string theory encompasses such a diaspora of mathematical concepts and integrates them in such a coherent way that most people tend to believe that those problematical issues can be resolved.

One of the many surprising features of string theory is that it requires ten or eleven dimensions to be mathematically consistent. It also makes great use of a conjectured property of space-time called supersymmetry, which relates matter particles known as fermions with force carrying particles called bosons. It is the question of how to break this extra symmetry and reduce the number of extra dimensions to describe the macroscopic world we live in that has been one of the outstanding problems in string theory since its inception over thirty years ago. Whilst much progress has been made in the attempt to bridge the gap between theoretical advances in string theory and phenomenological predictions, there is yet much more to be done.

Ultimately, it is the remarkable and beautiful mathematical structures unveiled by string theory that propels enquiries into its properties. Although string theory attempts to unify the description of disparate forces in a relatively simple framework, it could well be that the delay in achieving concrete physical results is due primarily to the inherent difficulty of the enterprise itself.

A qualitative description of some of the salient features of string theory might be of use before a more technical exposition is started. Of obliged reference are the excellent books [3–7] which contain a much clearer and comprehensive introduction than can be achieved here. It turns out there is only one input parameter in string theory, from which, at least in principle, all other physical quantities could be derived. This is the string tension, which is given by:

$$T = \frac{1}{2\pi l_s^2} = \frac{1}{2\pi\alpha'}$$

with l_s being the characteristic string length and $\alpha' = l_s^2$ a parameter typically used in perturbative expansions. The string length can be taken to be somewhere from just above current particle accelerator energy scales to all the way to the Planck length.

There are two basic types of strings in string theory: open strings which have end-points and closed string which do not. From these basic building blocks, there were found to be five different consistent ten-dimensional (super)string theories (as well as eleven-dimensional supergravity). Fortunately, if the theory aims to be the unique fundamental description of Nature, these different types of string theories were found to be related. Various dualities connected them, thus revealing that the string theories merely appeared different and were in fact all linked. So from this point of view, none of them is more fundamental than the rest.

As well as strings, it was realised that string theory also contains extended solitonic objects called Dirichlet p-branes [8], or Dp-branes for short. This refers to the fact that they can be defined as p-dimensional hypersurfaces on which open strings can end. They turn out to be incredibly useful, since they are also dynamical objects and are sources for Ramond-Ramond charges. To be more precise, a Dp-brane will couple to a (p+1)-form corresponding to the Ramond-Ramond antisymmetric tensor field $A_{(p+1)}$. In fact, a Dp-brane has a (p+1)-dimensional gauge theory living on its worldvolume whose fields are independent of the embedding space.

The dualities uncovered between the different string theories strongly suggested the existence of an eleven-dimensional theory - known as M-theory - whose low energy limit is eleven-dimensional supergravity. Although very little is known about high energy M-theory, including its dynamical degrees of freedom, it is now widely accepted that various strong/weak coupling limits link all the string theories to M-theory.

One way of studying the properties of M-theory is to look at its low energy approximation, eleven-dimensional supergravity [9]. This supergravity theory does not actually contain strings but rather membranes called M2-branes [10] and their Hodge dual partners, the M5-branes [11]. These objects are stable and therefore can survive the passage into the full quantum theory where any clues we might find could help us build a better picture of M-theory. It is also interesting to note that these M-branes descend into most of the known Dp-branes of the different string theories at different limits. Of course, since supergravity is also related to the rest of the string theories, analysing their properties is also important. In this thesis we shall be focusing mainly on eleven-dimensional supergravity and its dimensional reduction on a circle, Type IIA supergravity.

Another interesting consequence of the discovery of branes is the duality between string theory and certain gauge theories which is known as the AdS/CFT correspondence [12], or the gravity-gauge theory correspondence. This is the most sophisticated example of what is known as the holographic principle [13], which roughly says that any quantum theory of gravity should have a non-gravitational representation of its degrees of freedom in one lower dimension. In particular, many examples of branes wrapped on various manifolds and their dual Yang-Mills description have been analysed. In fact, the important issue of finding supergravity duals of realistic QCD-like theories is useful at least in understanding universal features of QCD [14]. In this thesis we shall look at branes wrapped on various supersymmetric cycles and analyse relevant field theory parameters.

Connected to the issue of finding and classifying supersymmetric supergravity solutions is the G-structures program [15], which in essence provides the form of the supergravity field strength, up to some undetermined components, from knowledge of the Killing spinors preserved by the background. This is a very powerful method which has proven useful both for deriving supergravity solutions as well as the classification of M-theory structure groups. In this thesis, we shall look at both of these facets and explore applications of these techniques.

1.1 Supergravity

We begin by reviewing the main aspects of eleven-dimensional supergravity, which arises as the low energy limit of M-theory, and Type IIA supergravity, which is the dimensional reduction of eleven-dimensional supergravity down to ten dimensions. We will briefly consider the particle spectrum and action of these theories.

1.1.1 Eleven-dimensional supergravity

The particle content can be obtained by studying the irreducible representations of the super-Poincaré algebra in flat eleven-dimensional spacetime, including topological terms:

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = (C\Gamma^M)_{\alpha\beta}P_M \pm \frac{1}{2}(C\Gamma_{MN})_{\alpha\beta}Z^{MN} \pm \frac{1}{5!}(C\Gamma_{MNPQR})_{\alpha\beta}Z^{MNPQR} \quad (1.1)$$

where the supersymmetry generators Q_α form a 32-component Majorana spinor, P_M is the momentum operator and the Z's are central charges. The spectrum contains a graviton G_{MN}, a three-form gauge potential A_{MNP} and a gravitino ψ^M_α. The existence of central charges allows for extended massive objects to have supersymmetric ground states. This is because they are a topological term that depend only on the homology class of the configuration. We can hence deduce that there are allowed membranes, called M2-branes, and also their Hodge duals, the M5-branes, which couple to these central charges. We note, however, that there is no allowance made for strings to exist in this theory. Also, as we shall see, there is a generalisation of the algebra to arbitrary supersymmetric backgrounds and worldvolume fields.

Eleven-dimensional supergravity is special because it is both minimal, in that it contains only one supersymmetry generator, and maximal, in that it is the highest spacetime dimension which admits massless supersymmetric multiplets corresponding to only particles with helicity $\Lambda \leq 2$. We will usually be interested in situations where the gravitino is set to zero, and look only at the bosonic fields G_{MN} and A_{MNP}. The action of these fields is given by

$$S = \frac{1}{2\kappa^2}\int d^{11}x\,\sqrt{-G}R - \frac{1}{2}\int\left(F\wedge *F + \frac{1}{3}A\wedge F\wedge F\right) \tag{1.2}$$

where $F = dA$ is the four-form field strength, $G = \det G_{MN}$ and R is the eleven-dimensional Ricci scalar. We have also indicated with a $*$ the Hodge duality operation. The quantity κ is related to the eleven-dimensional Newton constant G_{11} by

$$2\kappa^2 = 16\pi G_{11}.$$

There are alternative formulations of the action where the three-form A_{MNP} and its Hodge dual the six-form C_{MNPQRS} are treated in a more symmetric way, as well as considering the addition of source terms for the M2 and M5-branes [16], but we won't need to consider them here.

1.1.2 Type IIA supergravity

Now we can consider what happens when we dimensionally reduce the above action. The precise ansätze for the Kaluza-Klein reduction on a circle from eleven to ten dimensions is typically given by

$$ds_{11}^2 = e^{-\frac{2}{3}\phi} ds_{10}^2 + e^{\frac{4}{3}\phi}(R_{11}\psi + A_\mu dX^\mu) \tag{1.3}$$

where ψ has period 2π and $X^7 = R_{11}\psi$. We use X^7 as the eleventh dimension in order to be consistent with later work, even though this notation may be cumbersome on occasion. The scalar field ϕ is called the dilaton. We denote the full eleven-dimensional co-ordinates by Latin capital letters MNP and the strictly ten-dimensional fields by lower case Greek letters $\mu\nu\lambda$. If we take the X^7 co-ordinate to be a circle of radius R and take the $R \to 0$ limit, we find that the eleven-dimensional fields G_{MN} and A_{MNP} split into different components. In particular, they give rise to a metric $G_{\mu\nu}$, a one-form $A_\mu = G_{\mu 7}$ and the dilaton $\Phi = G_{77}$, as well as a three-form $A_{\mu\nu\lambda}$ and a two-form $B_{\mu\nu} = A_{\mu\nu 7}$.

This should coincide with the particle spectrum obtained from looking at the ten-dimensional supersymmetry algebra. In turn, this is derived from the irreducible representations of the super-Poincaré algebra in flat eleven-dimensional spacetime by splitting the 32-component Majorana spinor $\mathcal{Q}_\alpha$ into two Majorana-Weyl spinors. It turns out that the two resulting spinors have opposite chirality and we end up with Type IIA supergravity. The ten-dimensional algebra for Type IIA then takes the form

$$\begin{aligned} \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} &= (C\Gamma^M)_{\alpha\beta} P_M \pm \frac{1}{2}(C\Gamma_{MN})_{\alpha\beta} Z^{MN} \pm \frac{1}{5!}(C\Gamma_{MNPQR})_{\alpha\beta} Z^{MNPQR} \\ &\quad \pm (C\Gamma_{11})_{\alpha\beta} Z \pm (C\Gamma_M\Gamma_{11})_{\alpha\beta} Z^M \pm \frac{1}{4!}(C\Gamma_{MNPQ}\Gamma_{11})_{\alpha\beta} Z^{MNPQ} \end{aligned} \tag{1.4}$$

We can see that the spectrum coincides with that obtained from dimensional reduction. We note that there is now a one-form C_1 and a three-form C_3 in the spectrum of bosonic fields (and their Hodge dual five-form C_5 and seven-form C_7). These give rise to extended objects that are charged under these gauge potentials. In particular, these objects are known as D-branes, and Type IIA contains D2-, D4-, D6- and

D8-branes. In fact, there are also D0-branes which are interpreted as the dimensional reduction of the momentum modes along the M-theory circle (which we have denoted by X^7). Finally, there is also the fundamental string F1, equivalent to the dimensional reduction of the M2-brane, and the NS5-brane, which can be thought of as arising from the reduction of an M5-brane that does not lie along the direction X^7. We shall find all of these branes useful (with the exception of the D8-brane) throughout this thesis.

If we now look at the bosonic part of Type IIA supergravity action, we find that in the string frame it is given by

$$\begin{aligned} S_{IIA} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g_{10}}\left\{e^{-2\Phi}\left[R+4(\nabla\Phi)^2-\frac{1}{12}(H^3)^2\right]\right. \\ &\quad \left. -\frac{1}{4}(G^2)^2-\frac{1}{48}(G^4)^2\right\}-\frac{1}{4\kappa_{10}^2}\int B\wedge F\wedge F. \end{aligned} \tag{1.5}$$

The string frame is related to the Einstein frame by $G^E_{MN} = e^{-\phi/2}G_{MN}$ since this recovers the usual Einstein-Hilbert form of the part of the action involving the Ricci scalar R. We have denoted by g_{10} the ten-dimensional metric determinant, Φ is the dilaton, $H^3 = dB$ is the field strength of the NS-NS two-form B, while the R-R field strengths are $G^2 = dC_1$ and $G^4 = dC_3 + H^3 \wedge C_1$. Furthermore, we have that κ_{10} is related to the ten-dimensional Newton constant G_{10} by the relation $2\kappa_{10}^2 = 16\pi G_{10} = (2\pi)^7 g_s^2 l_s^8$, and g_s is set by the asymptotic value of the dilaton at infinity: $g_s = e^{\phi_\infty}$. This of course coincides with the Kaluza-Klein reduction of the eleven-dimensional supergravity action.

1.1.3 Basic objects: branes

As we have previously mentioned, these supergravity theories contain dynamical extended objects called branes. In both eleven-dimensional and Type IIA supergravity, they are p-dimensional extended objects that are charged with respect to a $(p+1)$-form gauge potential appearing in the supersymmetry algebra of the respective theory. All these branes will be BPS objects [17], which refers to the fact that they saturate a bound which relates their mass to their charge. Since their charge is of a topological origin, they are stable objects even when we go to high energies and into the unknown realm of M-theory. This condition also implies they are supersymmetric objects, and, in fact, flat branes typically preserve $\frac{1}{2}$ supersymmetry. They are thus commonly referred to as $\frac{1}{2}$-BPS states.

We will typically consider branes from two different points of view. On the one hand, the branes distort the spacetime around them, and so we can consider what particular configuration of branes source a certain supergravity background. In addition, they are also very useful as probes of a fixed background. Under appropriate conditions the backreaction of the probe brane on the geometry can be neglected. This technique takes advantage of the fact that all branes have a gauge theory of some description living on their worldvolume. In general, the gravitational surroundings of the probe brane, ie the background geometry, will have a gauge theory interpretation on the probes' worldvolume. We shall examine these issues more closely throughout this thesis.

As we have seen, low energy M-theory contains supersymmetric solutions called M2- and M5-branes. They are electrically charged under the field strengths F_4 and F_7 respectively. Many other BPS configurations can be constructed from these. Concretely, the supergravity solution corresponding to a stack of N parallel coincident M2- or M5-branes is given by [10, 11]

$$
\begin{aligned}
ds^2 &= H^{\frac{p-8}{9}} dx^2_{(1,p)} + H^{\frac{p+1}{9}} dx^2_{(10-p)} \\
F_{(p+2)} &= \mp d(H^{-1}) \wedge \epsilon_{(1,p)} \\
H &= 1 + \frac{c_p N}{r^{8-p}}
\end{aligned} \tag{1.6}
$$

where $p = 2, 5$ and c_p is a constant. The $\mp$ depends on the orientation of the branes and H is a harmonic function on the transverse space, with radial co-ordinate r given by

$$
r^2 = \sum_{i=p+1}^{(10)} (x^i)^2. \tag{1.7}
$$

We also denote the $(p+1)$-dimensional Minkowski part of the metric $ds^2_{(1,p)}$ by

$$
dx^2_{(1,p)} = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^p)^2 \tag{1.8}
$$

and $\epsilon_{(1,p)}$ is the volume form on this space. The totally transverse space metric $dx^2_{(10-p)}$ is simply

$$dx^2_{(10-p)} = (dx^{p+1})^2 + \cdots + (dx^{(10)})^2. \quad (1.9)$$

We note that the branes have a $(p+1)$-dimensional Poincaré invariant worldvolume and the $(10-p)$-dimensional transverse space has rotational invariance, and these are the expected spacetime symmetries. These equations (1.6) specify the full bosonic content of the supergravity solution for M-branes.

In order to recover the known D-brane spectrum of Type IIA supergravity, we need to compactify from eleven to ten dimensions on a circle. We note that we have two types of M-branes, those that are transverse to the S^1 we are compactifying on, and those that are wrapped on it. We find that if we dimensionally reduce M2- and M5-branes that are wrapped on this S^1 we recover the fundamental string F1 and the D4-brane (technically, the M5-brane worldvolume action reduces to the dual D4-brane action, as we shall see later). Considering M-branes which do not wrap the S^1, we find they reduce to D2-branes and NS5-branes respectively. Along with the D0-branes (which correspond to momentum modes along the S^1) and the D8-branes (which are slightly special [18, 19]) these reproduce the Type IIA brane spectrum. Concretely, the Type IIA supergravity solution corresponding to a stack of N parallel coincident Dp-branes (in the string frame) is given by

$$\begin{aligned} ds^2 &= H^{-1/2}dx^2_{(1,p)} + H^{1/2}dx^2_{(9-p)} \\ G^{(p+2)} &= \mp d(H^{-1}) \wedge \epsilon_{(1,p)} \\ e^{\phi} &= H^{\frac{3-p}{4}} \\ H &= 1 + \frac{c_p N}{r^{7-p}}. \end{aligned} \quad (1.10)$$

As before, this solution describes branes situated at $r = 0$ with worldvolume coordinates $x^0, \ldots, x^p$. The notable difference from eleven-dimensional supergravity is the presence of the dilaton. As before, the harmonic function H determines the brane solution. For Type IIA, only the values $p = 0, 2, 4, 6, 8$ are allowed.

As can be seen from the Type IIA action, however, there can also be branes which are charged with respect to the NS-NS three-form $H^{(3)}$. These are the fundamental string and its Hodge dual the NS5-brane. We will not be particularly interested in these solutions for our purposes, so will not present them here (interested readers may check [7] for these solutions).

The solutions presented above are for simple parallel coincident configurations of branes, which all preserve $\frac{1}{2}$ supersymmetry. There are also more complicated configurations, such as: intersecting branes, branes which are wrapped on various manifolds, and branes which end on other branes. We shall look at examples of these interesting configurations and the process of obtaining supergravity solutions in more detail later on. These are all examples of branes warping the spacetime around them and sourcing background geometries. We now turn to a brief review of branes as probes of the geometry.

1.2 Branes as probes

One of the most fruitful techniques in string theory has been the use of branes as probes of supergravity backgrounds. The idea is to place a "test" probe brane in a fixed supergravity background and then examine its dynamics. In order for this to make sense, we must work in the approximation that the backreaction of the brane on the background is negligible. Since we will be interested in BPS configurations of branes and also supersymmetric probes, this approximation is well justified.

One of the advantages of this technique is that many properties of gauge theories can be understood geometrically. This connection arises because the branes are both dynamical objects which cause gravitational perturbations in their surroundings, and the fact that these dynamics can be described through a worldvolume action, which we now introduce.

As previously mentioned, the key ingredient used in probe calculations is the worldvolume action. If we consider the case of Dp-branes, and restrict to the bosonic fields, we find that their worldvolume action is given by the $(p+1)$-dimensional Dirac-Born-Infeld (DBI) action together with Wess-Zumino couplings:

$$S_{DBI}^{(p+1)} = -T_{D_p}\left\{\int d^{p+1}\sigma\, e^{-\phi}\sqrt{-\det(G_{mn}+\mathcal{F}_{mn})} - \int \sum_n \mathcal{P}(C_{(n)}) \wedge e^{\mathcal{F}}\right\}, \quad (1.11)$$

where T_{D_p} is the tension of the Dp-brane given by $T_{D_p} = \frac{1}{(2\pi)^p g_s l_s^{p+1}}$ and where $\mathcal{F} = 2\pi l_s^2 F - \mathcal{P}(B)$ is a linear combination of the pullback of the spacetime NS-NS two-form potential B and a worldvolume two-form $U(1)$ field strength F. The pullback of the spacetime metric onto the brane is also understood. The second term is a compact way of writing the couplings of the R-R fields and the sum is over $n = 1, 3, 5, 7, 9$ with the integral being restricted to include only $(p+1)$-forms (natural since we are integrating over the $(p+1)$-dimensional brane worldvolume). These Wess-Zumino couplings give rise to interesting field theory phenomena such as instanton and monopole states, which we examine in more detail in later sections.

If we consider single branes in flat space, for example, the low energy limit of this action in the absence of a background B field is given by the $(p+1)$-dimensional $U(1)$ maximally supersymmetric Yang-Mills action. More general setups with non-trivial B field are described by non-commutative field theories [20]. We will not be considering those here. If we now fix the reparameterisation invariance of the worldvolume by setting $\sigma^0 = x^0, \sigma^1 = x^1, \ldots, \sigma^p = x^p$ then the directions transverse to the brane are $x^{p+1}, \ldots, x^9$. We can interpret these as scalar fields on the worldvolume action

$$\Phi^i(\sigma^a) = \frac{1}{2\pi l_s^2}\, x^i(\sigma^a) \quad (1.12)$$

where $i = p+1, \ldots, 9$. Expanding the DBI action for small l_s in flat space and looking at the leading term in the expansion we find

$$S_0 = -4\pi^2 l_s^4 T_{D_p} \int d^{p+1}\sigma \left(\frac{1}{2}\partial_a \Phi^i \partial^a \Phi^i + \frac{1}{4} F_{ab}F^{ab}\right). \quad (1.13)$$

The Yang-Mills coupling can be read off from these expressions to be

$$\frac{1}{g_{YM}^2} = 4\pi^2 l_s^4 T_{D_p}. \quad (1.14)$$

If we consider the more general case of N coincident Dp-branes, then the worldvolume action should be non-Abelian. The precise nature of this action to all orders is still not well known. However, in the limit $l_s \to 0$ the low energy dynamics are given by a maximally supersymmetric $U(N)$ Yang-Mills theory. The main point of this description is that field theory states and gauge groups, for example, can be identified geometrically with certain string and brane configurations.

1.3 The AdS/CFT correspondence

The AdS/CFT correspondence [12] describes a duality between string or M-theory and gauge theories. It is a concrete realisation of the older idea of the holographic principle. As with the rest of the topics covered in this introduction, there are many introductory reviews on this subject [21], as well as a large body of research, so we will not go into much detail here. Although this duality has not yet been proved in a mathematical sense, there is a large body of evidence which supports it.

It was originally inspired by the properties of D3-branes in Type IIB supergravity. This supergravity is the T-dual of Type IIA, that is, the Majorana-Weyl spinors that act as its supersymmetry generators have the same chirality. In any case, this is the best-studied example and we proceed to illustrate it.

Considering a stack of N coincident D3-branes in Type IIB supergravity, from the previous section we know that this will have a worldvolume description in terms of an $\mathcal{N} = 4$ four-dimensional $SU(N)$ Yang-Mills theory. One may now ask what the supergravity description of this brane configuration could be. Now we recall that to find the field theory description we took the $l_s \to 0$ limit, which effectively decouples gravity modes and massive string states. However, any sensible limit should keep certain field theory quantities such as the gauge coupling and massive states fixed. This requires that we keep the 't Hooft coupling

$$g_{YM}^2 N = 2\pi g_s N \tag{1.15}$$

and the gauge masses (which scale as $m \sim r/l_s^2$) fixed while taking $l_s \to 0$. It turns out that the correct quantity to fix is given by $U \equiv r/l_s^2$, where r is the radial coordinate transverse to the branes. This implies that we are effectively also taking the $r \to 0$ limit, which means we are considering the region close to the branes. This limit is usually referred to as the near-horizon limit for this reason. In terms of this variable U, the harmonic function H which specifies the solution of the D3-brane becomes

$$H = 1 + \frac{4\pi N g_s l_s^4}{r^4} \rightarrow \frac{4\pi N g_s}{U^4}. \tag{1.16}$$

In this limit the spacetime metric becomes

$$\frac{1}{l_s^2} ds^2 = \frac{U^2}{L^2} ds^2_{(1,3)} + L^2 \frac{dU^2}{U^2} + L^2 d\Omega_5^2 \tag{1.17}$$

where $L^2 = \sqrt{2g_{YM}^2 N}$. This metric describes an $AdS_5 \times S^5$ spacetime with L as the radius of both the AdS_5 and the S^5. The regions of validity of this correspondence are small curvature on the supergravity side and small 't Hooft coupling $\lambda = g_{YM}^2 N$ on the gauge theory side. These are opposite regions of validity since small curvature on the supergravity side implies large radius L and a correspondingly large 't Hooft coupling. Therefore, in principle, by considering the supergravity approximation we can make predictions about the non-perturbative nature of the gauge theory.

This correspondence is the main motivation for finding supergravity solutions that are dual to interesting field theories. It has been extended to more general brane configurations with reduced supersymmetry which are dual to a variety of Yang-Mills theories [22, 23]. We shall look at some examples of wrapped branes and their dual field theory description in later sections.

1.4 Outline of thesis

The main aim of this thesis is to investigate various geometric and field-theoretic properties of wrapped M5-branes preserving either 16 or 8 real supersymmetries. Supersymmetric brane probes are used extensively in this context, making full use of their dual nature as dynamic gravitational objects which contain a gauge theory on their worldvolume. We begin Chapter 2 by discussing methods of finding eleven-dimensional supergravity solutions of intersecting M5-brane configurations. The method of solving the Killing spinor equation to arrive at supersymmetric solutions of interest is briefly reviewed. The connection with the G-structures program of classifying supersymmetric solutions and the bilinear spinor formalism employed is also summarised. The Type IIA Hanany-Witten models, which are the dimensional reduction of the intersecting M5-brane solutions, are also introduced. The four-dimensional worldvolume theory of the Hanany-Witten construction is outlined, and reference made to the Seiberg-Witten analysis. The explicit form of the M5-brane action is presented and its general features outlined. Finally, we also include a short discussion on the method of calibrations, which involves finding forms that are useful in determining supersymmetric surfaces that have minimal volume in their homology class. This is connected to minimal cycles that branes will be permitted to wrap in order to preserve supersymmetry. We also give a short outline of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ gauge theories. These topics will the contents of Chapter 2.

In Chapter 3, we look at various supersymmetric probes of the supergravity backgrounds we have reviewed in the previous chapter. Both M2- and M5-brane probes are used to calculate certain parameters of the dual gauge theory. In particular, M2-brane probes are employed to determine the BPS spectrum of the $\mathcal{N} = 2$ dual gauge theory in the first example, and M5-branes are used to calculate field theory parameters such as the gauge coupling, the theta angle and the complex scalar moduli space metric. The instanton action is calculated using a D0-brane probe once the limit to Type IIA is taken. Finally, an alternative method of deriving the $\mathcal{N} = 1$ supergravity solution involving the projection conditions and the bilinear spinor formalism is illustrated.

In Chapter 4, we deepen our discussion of calibrations and consider the underlying structure of supersymmetric backgrounds by examining their superalgebras. We consider the most general form of the superalgebra for eleven-dimensional supergravity and establish some of the allowed brane probes for both backgrounds under discussion. We discuss the relevance to G-structures in a little more detail and also introduce the notion of structure groups of supersymmetric M-theory backgrounds. These have been classified for the most part, and consist of finding the subgroups of the isotropy group of a single Killing spinor in eleven-dimensional supergravity. The Killing vector built from this spinor turns out to be either time-like or null, which leads to different classes of structure groups. These are shown to be illustrated by our brane configurations with some hindsight, and provide interesting field-theoretic and geometric interpretations of the supergravity backgrounds under discussion. Finally, some field theoretic objects such as: instantons, monopoles, vortices and domain walls are briefly introduced, and their geometric interpretation in terms of branes described.

In Chapter 5, we make use of the concepts we have learnt in the last chapter and analyse the central charges of the two supergravity backgrounds we have been discussing. This reveals what supersymmetric probes are allowed in this background, and also ties in well with the geometric structure groups which are allowed in terms of the number of Killing spinors the background preserves. All these ingredients allow us to identify calibrating forms in these backgrounds. The integral of these calibrating forms have a field theory interpretation as the tension of BPS vortices, domain walls and monopoles. Such objects are shown to be supersymmetric and stable from this perspective.

Finally, Chapter 6 concludes and summarises the main results of this thesis and considers what possible avenues of research are available for the future.

Chapter 2

Supergravity solutions

In this chapter any reference to supergravity solutions refers to eleven-dimensional supergravity unless otherwise indicated. We introduce the requirements that a solution be supersymmetric in terms of satisfying the Killing spinor equation. We will make the relation between intersecting branes of the same type and branes wrapping smooth cycles clear, and express the supersymmetry preserving conditions in terms of calibrations.

2.1 Supergravity solutions of intersecting branes

One method of finding the supergravity solution of a particular brane configuration consists of using the projection conditions for preservation of supersymmetry to constrain the metric and four-form field strength.

The integrability of the Killing spinor equation then allows one to relate geometries that preserve some degree of supersymmetry to those that in addition solve the equations of motion [24]. Typically, a supersymmetric solution which in addition solves the Bianchi identity and equations of motion for the background field strength will satisfy almost all the constraints from the Einstein equations. For the case of a geometry admitting a null Killing vector $K = e^+$, for example, one just needs to impose $E_{++} = 0$ to obtain a full supersymmetric solution [24], where $E_{\mu\nu} = 0$ refers to the Einstein equations.

Recently, a great deal of progress has been made in understanding the general structure of supersymmetric solutions of supergravity theories [24–29]. For the case of eleven-dimensional supergravity, the classification of the local form of supersymmetric solutions of a background preserving one Killing spinor has been completed [24, 29]. This is a group theoretic approach which has proven useful in classifying supersymmetric solutions in various dimensions, depending on whether the minimal background Killing spinor gives either a time-like or null Killing vector. We shall not go into the details of this approach here, but we note that both supergravity solutions used in this paper can also be derived from the G-structures approach [30]. Quite similar to this approach is the new method we employ to derive the $\mathcal{N}=1$ supergravity solution [31] in the first part of Chapter 3. For the moment we consider the original approach of solving the Killing spinor equation directly.

For a general supergravity solution of a certain brane configuration, and since we are only considering purely bosonic solutions for simplicity, we must check that the supersymmetry transformations of the fermionic fields which vanish are those that are preserved by the solution.

To illustrate this, we consider the fermionic field of supergravity ψ, the spin-3/2 Rarita-Schwinger field. For a supersymmetric bosonic solution we must ensure that the supersymmetry variations of this field vanish. This leads us to what is referred to as the Killing spinor equation:

$$\tilde{D}_I \epsilon = 0, \tag{2.1}$$

where

$$\tilde{D}_I \epsilon \equiv \nabla_I \epsilon + \frac{1}{288}\left[\Gamma_I{}^{JKLM} - 8\delta_I^J \Gamma^{KLM}\right] F_{JKLM}\epsilon \tag{2.2}$$

and

$$\nabla_I = \partial_I + \frac{1}{4}\omega^n_{\ p\,I}\hat{\Gamma}^p_{\ n}. \tag{2.3}$$

We use the notation in [32], with the spin connection denoted by $\omega^n_{\ p\,I}$ (notice this has mixed indices acting on both the tangent space and spacetime), F denotes the four-form field strength, and Γ_M for the spacetime Dirac gamma-matrices and $\hat{\Gamma}_m$

for the tangent-space gamma-matrices. These are related by the vielbein e_M^m such that

$$g_{MN} = e_M^m e_N^n \eta_{mn} \ , \ \Gamma_M = e_M^n \hat{\Gamma}_m \ , \ \{\Gamma_M, \Gamma_N\} = 2g_{MN} \ , \ \left\{\hat{\Gamma}_m, \hat{\Gamma}_n\right\} = 2\eta_{mn}. \tag{2.4}$$

The condition $\tilde{D}_I\epsilon = 0$ on the thirty-two component spinor ϵ amounts to a set of linearly independent constraints, each of which must be put to zero independently. Since the number of non-trivial components of ϵ determines the amount of supersymmetry, a solution that possesses N independent Killing spinors equivalently preserves N out of a possible 32 supersymmetries. If the Killing spinor equation is satisfied for a given metric, and the resulting four-form obeys the constraints $dF = 0$ and $d * F = 0$, then most components of Einstein's equations are guaranteed to be satisfied. This then specifies the bosonic content of a BPS supergravity solution.

The number of supersymmetries preserved by a p-brane configuration is given by the number of spinors ϵ which satisfy the equation

$$\hat{\Gamma}\epsilon = \epsilon \tag{2.5}$$

where we have the definitions $\hat{\Gamma} = \frac{1}{p!}\epsilon^{\alpha_1\cdots\alpha_p}\Gamma_{M_1\ldots M_p}\partial_{\alpha_1}X^{M_1}\cdots\partial_{\alpha_p}X^{M_p}$ and $\Gamma_{M_1\cdots M_p} = \frac{1}{p!}\Gamma_{[M_1\cdots M_p]}$. The X^M is the embedding of the p-brane in the background geometry and the α_i denote the worldvolume coordinates.

The M-brane supergravity solutions we have so far considered have the important property that they preserve $\frac{1}{2}$ supersymmetry. Concretely, if we consider a flat M5-brane with worldvolume $X^{\alpha_0}\cdots X^{\alpha_5}$ (which in shorthand we will write 012345), then this object preserves 16 out of a possible 32 real supersymmetries corresponding to the components of a spinor η which satisfies the condition

$$\hat{\Gamma}_{012345}\eta = \eta. \tag{2.6}$$

More BPS states can be built from multiple M-branes by combining them in such a way that some amount of supersymmetry is preserved. For a general p-brane, and considering the case with no worldvolume fields other than scalars turned on, there is a general caveat that each pair share a $(p-2)$-dimensional spatial intersection. This is because all p-branes contain a $(p-1)$-dimensional gauge field A_{p-1} (which is the worldvolume dual of the scalar field) which allows for a dynamical self intersection. The Killing spinors of the resulting configurations are those that survive the complete set of projection conditions.

However, M5-branes include a two-form on their worldvolume which allows for a dynamical one-dimensional spatial intersection of two M5-branes. These intersections are much less understood even in the context of string theory, where, for example, they can be dimensionally reduced to two D4-branes intersecting over a point or T-dualised to an intersecting D0-D8-brane system [33,34]. In any case, we shall consider these types of intersections in more detail in Chapter 5 when we look at central charges of wrapped M5-brane backgrounds.

For the time being, we specialise to $(p-2)$-dimensional intersections of M5-branes. The simplest example we can look at is the case of orthogonal intersections of flat M5-branes. We can consider, for definiteness, an example of two sets of orthogonally intersecting M5-branes with worldvolume directions 012345 and 012367. This corresponds to the projection conditions

$$\hat{\Gamma}_{012345}\epsilon = \epsilon \tag{2.7}$$

$$\hat{\Gamma}_{012367}\epsilon = \epsilon. \tag{2.8}$$

Now, because these commute with each other, it can be shown that one quarter supersymmetry is preserved. In general, for a configuration of m types of intersecting branes to preserve some supersymmetry, a necessary condition is that each pair of branes must preserve a quarter supersymmetry, like in our example. The whole configuration will then preserve a minimum of $1/2^m$ supersymmetry, depending on whether some projection conditions are redundant or are already implicitly imposed by the others. This occurs when some projection conditions are not just a traceless product of $\hat{\Gamma}$-matrices but rather when some product of these operators is plus or minus the identity.

As we can show, there are alternative ways to write these conditions. For example, the above projection conditions imply that

$$\hat{\Gamma}_{4567}\epsilon = \hat{\Gamma}_{012389(10)}\epsilon = -\epsilon. \tag{2.9}$$

where we have used the identity $\hat{\Gamma}_{0123456789(10)} \equiv 1$. This last condition is the projection condition for a KK6-brane with supersymmetry preserving orientation prescribed by the orientation of the original M5-branes. When reduced dimensionally, a KK6-brane, which is pure geometry in eleven dimensions, becomes a D6-brane, hence the name. So even though the presence of this extra brane is implied in our original projection conditions, it does not break any further supersymmetry, and the whole system still preserves one quarter supersymmetry.

Reducing this system to Type IIA along the direction x^7, we would find a quarter BPS system of orthogonally intersecting NS5-branes [012345], D4-branes [01236] and D6-branes [012389(10)]. This system is called a Hanany-Witten model [35] and we shall describe it in more detail promptly.

Now the aim of this introduction is to make the connection between intersecting branes of the same type and a brane wrapping a smooth cycle. We also need to show how the conditions for preservation of supersymmetry can be expressed geometrically in terms of calibrations. To this end, since we are going to be working with holomorphic cycles in a target space with a complex structure, it makes sense to re-write the Clifford algebra in terms of complex co-ordinates. If we now define complex co-ordinates

$$z^1 = v = x^4 + ix^5 \tag{2.10}$$

$$z^2 = s = x^6 + ix^7, \tag{2.11}$$

and complex gamma matrices

$$\hat{\Gamma}_{z^1} = \frac{1}{2}\left(\hat{\Gamma}_{x^4} - i\hat{\Gamma}_{x^5}\right), \tag{2.12}$$

for example, then we can concisely express the above relations as

$$\hat{\Gamma}_{0123a\bar{b}}\epsilon = i\delta_{a\bar{b}}\epsilon \tag{2.13}$$

with $\delta_{a\bar{b}}$ $(a, b = 1, 2)$ as the tangent-space metric with $\delta_{1\bar{1}} = \frac{1}{2}$ in our conventions. We also use the shorthand $\hat{\Gamma}_{z^a} = \hat{\Gamma}_a$ notation. This restriction on ϵ means that the solution will preserve $\frac{1}{4}$ of the supersymmetry (or equivalently, eight supercharges), which corresponds to $\mathcal{N} = 2$ in four dimensions. We use conventions where $ds^2 = 2g_{M\bar{N}}dz^M dz^{\bar{N}} = \delta_{a\bar{b}}e^a_M \left(\overline{e^b_N}\right) dz^M dz^{\bar{N}}$ for complex Hermitian metrics.

Now we can easily check that we can add an M5-brane with embedding defined by an arbitrary holomorphic curve without breaking any more supersymmetry. We do this by embedding the M5-brane in the 4567 directions as the zeroes of a holomorphic function $f(v, s)$. We can take this curve to be, for example, $f(v, s) = vs - c = 0$ in $\mathbf{C}^2$, where c is a constant. In the limiting case where $c = 0$ and the curve becomes singular, the function f factorises and describes a system of two orthogonally intersecting M5-branes spanning the v and s planes respectively. Thus, a system of two orthogonally intersecting M5-branes, such as the one we started out with, can be thought of as the singular limit of an M5-brane wrapping a smooth two-cycle.

In general, a complex structure such as the one above can be defined on the relative transverse space (those directions which are common to some branes but not all) of a system of n intersecting branes, yielding the picture of a brane wrapping a smooth cycle in $\mathbf{C}^n$. The Killing spinors and associated supersymmetries of a wrapped brane of this kind must be the same as those in the singular limit described by orthogonal intersections of branes (since their holomorphic embedding curves differ only by a constant) and can be easily calculated.

2.2 Standard calibrations

There are other useful ways of understanding the geometry of supersymmetric brane configurations. The one which we will use throughout this thesis is the powerful method of calibrations [36–39]. This technique has been used to classify the supersymmetric cycles which branes can wrap in various special holonomy manifolds [40–42]. It allows one to find minimal energy configurations for probe branes in various backgrounds. It is also very useful in its generalised form when it comes to finding supergravity solutions and calculating central charges of the supersymmetry algebra of brane probes. Calibrations are also intimately connected with the notion of structure groups, which we shall discuss later as well.

For now, we introduce the concept of a calibrating form. As we will see, calibrations are p-forms which enable us to classify minimal p-dimensional submanifolds in a particular spacetime background. These surfaces have the property that they have minimal volume in their homology class. Thus, the problem of finding supersymmetric probes which minimise the energy is transformed to the more straightforward problem of finding calibrated surfaces for a particular background manifold. Calibrating forms of special holonomy manifolds have been extensively studied and classified [43]. We will be most interested in the Calabi-Yau and Hyper-Kähler cases in our investigations.

We begin by considering a d-dimensional manifold $(\mathcal{M}, g)$. The standard definition of a calibration $\phi \in \Lambda^p\mathcal{M}$ is that it satisfies

$$\int_{\mathcal{M}_p} \phi \leq vol(\mathcal{M}_p) \quad \text{and} \quad d\phi = 0 \tag{2.14}$$

where $\mathcal{M}_p$ is an arbitrary p-dimensional submanifold of $\mathcal{M}$. Since we are more interested in dealing with branes of infinite spatial extent, the precise definition we shall use is that a p-dimensional oriented submanifold $\mathcal{M}_p$ is called calibrated if at every point on $\mathcal{M}_p$, the pullback of ϕ to some tangent space $T_x\mathcal{M}_p$ is equal to the volume form on that tangent space,

$$\mathcal{P}_{T_x\mathcal{M}_p}\phi = vol(T_x\mathcal{M}_p). \tag{2.15}$$

Note that the condition for a calibrated submanifold is a local one.

For spacetimes with no background flux, there is a classification of the calibrations that can exist in these manifolds, called special holonomy manifolds. For the cases of interest to us, a manifold of special holonomy is indicated by the the existence of a covariantly constant spinor. Holonomy is a measure of the transformation of a field ψ upon parallel transport around a contractible closed curve. In the example of a Riemannian manifold $\mathcal{E}$ of dimension n, the spin connection is, in general, an $SO(n)$ field. As $\psi \rightarrow U\psi$ around the loop, the $SO(n)$ matrices U form the holonomy group H of the manifold. The manifold $\mathcal{E}$ is said to be of special holonomy when H

is a proper subgroup of $SO(n)$.

Now, for any general background, we know that supersymmetry preservation implies the existence of covariantly constant spinors, as typified, for example, in the Killing spinor equation. This is the fundamental reason why special holonomy manifolds are so useful when constructing supersymmetric supergravity solutions. Moreover, manifolds of special holonomy are useful because they naturally have calibrating forms associated to them, as we shall see.

It is well known that, in general, the holonomy of the background geometry is reduced once fluxes (or equivalently a non-trivial four-form field strength) are turned on. There is, in general, a torsion which modifies the usual connection on the manifold, thus giving it a reduced group structure. Group structures are more general since holonomy requires that certain differential conditions be satisfied (such as the Nijenhuis tensor vanishing). We shall discuss this more in depth when we consider G-structures.

For backgrounds with no flux, the classification of special holonomy manifolds has been done by Berger [43], and is based upon the classification of Lie groups. A brief synopsis of the main results are:

Calabi-Yau manifolds

One possibility for the special holonomy group is $H = SU(n) \subset SO(2n)$, where $d = 2n$ is the real dimension of the manifold under consideration. Since $n \in \mathbf{Z}$, this forces the manifold to be even dimensional. These manifolds are called Calabi-Yau, also denoted $CY(n)$ or Calabi-Yau n-fold. They are Kähler manifolds with vanishing Ricci form (or equivalently first Chern class). This implies that it admits at least two independent invariant forms which are nowhere vanishing. These are expressed in terms of the Kähler two-form $J = i\delta_{a\bar{b}}e^a \wedge e^{\bar{b}}$ and the holomorphic $(n, 0)$-form $\Omega_n = e^{z_1} \wedge \cdots \wedge e^{z_n}$ for a suitable choice of complex vielbein. These invariant forms are given by

$$\phi_1 = \frac{1}{p!}J^p \quad p \in 1, \ldots, n \tag{2.16}$$

$$\phi_2 = Re(e^{i\theta}\Omega). \tag{2.17}$$

In particular, the calibrating $2p$-forms $\phi_1 = \frac{1}{p!}J^p$ are known as Kähler calibrations. The form ϕ_2 includes an arbitrary constant phase θ owing to the $SU(n)$ structure giving a certain freedom to rotate complex structures. Clearly, both these forms are closed and also satisfy the second property for a calibration.

The calibrated submanifolds of a Calabi-Yau are thus complex submanifolds (by the Wirtinger theorem), which are calibrated by ϕ_1, and so-called Special-Lagrangian submanifolds, which are n-cycles calibrated by ϕ_2. It turns out that the fraction of preserved supersymmetry of a $CY(n)$ is $1/2^{(n-1)}$.

Hyper-Kähler manifolds

A further possibility for the holonomy group is $H = Sp(n) \subset SO(4n)$, where $d = 4n$ is the dimension of the manifold in question. Calabi-Yau 2-folds are automatically hyper-Kähler since $SU(2) \sim Sp(1)$ so we take $n \geq 2$ and an integer. These manifolds are usually denoted by HK_n and are a generalisation of a Kähler manifold in the sense that a family of complex structures is allowed. The complex structures can be parametrised by two-spheres S^2, with SU(2) commutation relations between them. In general, there will thus be a family of Kähler two-forms J^i and holomorphic $(4n, 0)$-forms, for i being a multiple of three. As before, these forms are closed and can be used to construct calibrating forms in the same way as the Calabi-Yau case.

We shall only be interested in the case of Calabi-Yau 2-folds (or HK_1) and in that particular case, there are just three inequivalent complex structures with SU(2) commutation relations between them. In that case, there are then three possible Kähler forms that can be constructed, giving the calibrated complex submanifolds. The special-Lagrangian submanifolds are simply holomorphic curves with respect to a different choice of complex structure. This example will be illustrated for the supergravity solution preserving 16 real supersymmetries, which we investigate in the next chapter.

Exceptional holonomy groups

Lastly, there are a couple of exceptional holonomy groups which arise in unique dimensions: in eight dimensions, when the holonomy group is $H = Spin(7) \subset SO(8)$, and in seven dimensions, when it is $H = G_2 \subset SO(7)$. Manifolds of G_2 holonomy contain an invariant three-form as well as its Hodge dual four-form, called associative and co-associative calibrations, respectively. These have been widely used in M-theory compactifications since they preserve a certain amount of supersymmetry, but we shall not be needing them here.

Of more interest is the eight-dimensional case given by a manifold of $Spin(7)$ holonomy. These are Ricci flat and contain covariantly constant spinors as well, and therefore are useful when constructing supersymmetric supergravity solutions. They contain a self-dual four-form Ψ_4 which is invariant under $Spin(7)$ and gives rise to a four-dimensional calibration called the Cayley calibration.

Having looked at the case of zero flux, we can ask what happens when we turn the flux on. In that case, the original Killing spinors are determined not only by the metric but also the field strength, and therefore in general not covariantly constant. In these cases, we can interpret the background field strengths as torsion owing to the way the Killing spinor equation is modified. So the Killing spinors will now be covariantly constant only with respect to this modified connection that includes the flux. From these one can construct what are now called generalised (since they include flux) calibration forms [38, 39, 44].

The existence of these Killing spinors and vectors with respect to the connection with torsion can be understood in terms of reduced holonomy groups or structure groups. One of the most interesting advances in recent years has been the classification of supersymmetric supergravity solutions by covariantly constant generalised calibrating forms, referred to as G-structures. Not only supergravity solutions in eleven-dimensions (which we will focus on here), but also the cases of four, five, six, seven and ten dimensions (see [15] and references therein).

As mentioned, recently all maximally supersymmetric solutions of supergravity in eleven dimensions have been classified. The formalism used, and one which we shall exploit throughout this thesis, is called the bilinear spinor formalism, which we briefly review.

2.3 G-structures: bilinear spinor formalism

A problem which has generated much interest in recent years has been the classification of the local forms of all supersymmetric solutions of supergravity theories. For the case of eleven-dimensional supergravity, a complete analysis has already been carried out [24, 29]. These solutions preserve at least one of the background Killing spinors. The approach used in those papers, and which we briefly review, is to construct p-forms of different degrees from the Killing spinor preserved by the background.

The basic idea is to construct the p-forms by defining, for example, the Killing vector $K_M = \bar{\epsilon}\Gamma_M\epsilon$ where the commuting Killing spinor ϵ and its hermitian conjugate $\bar{\epsilon} = \epsilon^T\Gamma_0$ have all spinor indices contracted. The term Killing spinor is then justified by the fact that we can use it to construct Killing vectors such as K. These p-forms are also referred to as spinor bilinears by the nature of their construction.

It is useful to realise that the spinor $\epsilon(x)$ which satisfies the Killing spinor equation for preservation of supersymmetry can be reconstructed (up to a sign) from the following one-, two- and five-forms built from spinor bilinears:

$$\begin{aligned} K_M &= \bar{\epsilon}\Gamma_M\epsilon \\ \Omega_{MN} &= \bar{\epsilon}\Gamma_{MN}\epsilon \\ \Sigma_{MNPQR} &= \bar{\epsilon}\Gamma_{MNPQR}\epsilon. \end{aligned} \tag{2.18}$$

One can check that the zero-, three- and four-forms built in a similar way vanish identically.

These spinor bilinears also satisfy a set of algebraic and differential relations. The algebraic relations can be derived from Fierz identities. The differential relations follow essentially from the Killing spinor equation, and, for example, one can show that the above one-form K is actually a Killing vector and satisfies $\nabla_{(m}K_{n)} = 0$.

The exterior derivatives of the forms are then given by

$$
\begin{aligned}
dK &= \frac{2}{3}\iota_\Omega F + \frac{1}{3}\iota_\Sigma * F && (2.19)\\
d\Omega &= \iota_K F && (2.20)\\
d\Sigma &= \iota_K * F - \Omega \wedge F. && (2.21)
\end{aligned}
$$

where the contraction is defined by $(\iota_\Omega F)_{MN} = (1/2!)\Omega^{AB}F_{ABMN}$.

Moreover, these p-forms constructed from the Killing spinors of the background satisfy the properties required for a calibration. Furthermore, static brane probes wrapping calibrated submanifolds specified by these p-forms are supersymmetric as well as volume minimising. This can be shown explicitly in a simple way.

Consider the p-form ϕ built from spinor bilinears, as above. Firstly, we notice straight away that since ϵ is covariantly constant, and we are presently considering the case of no background fluxes, the closure of ϕ is guaranteed. In order to check the second condition for a calibrating form, we pull back ϕ onto a tangent p-plane ζ, with local co-ordinates $\sigma^1, \ldots, \sigma^p$ to obtain

$$
\mathcal{P}_{T_\zeta}\phi = \epsilon^T\Gamma_\zeta\epsilon\sqrt{\tilde{\gamma}}d^p\sigma \tag{2.22}
$$

with the matrix Γ_ζ given by

$$
\Gamma_\zeta = -\frac{1}{\sqrt{\tilde{\gamma}}}\,\epsilon^{a_1\ldots a_p}\partial_{a_1}x^{i_1}\cdots\partial_{a_p}x^{i_p}\Gamma_{0i^1\ldots i^p}, \tag{2.23}
$$

and $a_1, \ldots a_p$ refer to the p-plane co-ordinates $\sigma^1, \ldots \sigma^p$. The factors of $\tilde{\gamma}$ refer to the determinant of the induced metric on the p-plane. Now, since Γ_ζ is hermitian and satisfies $\Gamma_\zeta^2 = 1$, the following inequality holds from the projection condition $\frac{1}{2}(1-\Gamma_\zeta)\epsilon = 0$:

$$
0 \leq \left\|\frac{1}{2}(1-\Gamma_\zeta)\epsilon\right\|^2 = \epsilon^T\frac{1}{2}(1-\Gamma_\zeta)\epsilon. \tag{2.24}
$$

We have used the fact that in the absence of any background field strengths ϵ is a covariantly constant spinor which we can normalise $\epsilon^T\epsilon = 1$. The inequality then reads

$$
\epsilon^T\Gamma_\zeta\epsilon \leq \epsilon^T\epsilon = 1. \tag{2.25}
$$

Looking back at (2.22) and using (2.25) we finally see that

$$\mathcal{P}_{T_\zeta}\phi \leq \sqrt{\tilde{\gamma}}d^p\sigma = vol(T_\zeta) \tag{2.26}$$

which is the second local condition required for a calibration.

It is important to note that the calibration bound (2.25) is saturated when $\Gamma_\zeta\epsilon = \epsilon$, which is exactly the projection condition for preservation of supersymmetry for a static p-brane probe. So we can conclude that the calibrated cycles of ϕ are also cycles on which probe branes can wrap and preserve supersymmetry. This construction of calibrating forms from spinor bilinears is quite general and extends to the case of generalised calibrations in a supersymmetric background with flux, as we shall see.

Moreover, these forms also define a mathematical structure called a G-structure. For eleven-dimensional supergravity, which deals with Lorentzian manifolds with a spin structure, we start with a $Spin(10,1)$ structure and a globally well-defined spinor which defines K, Ω and Σ. From these considerations, it was found that at a point, the isotropy group of the spinor is either $SU(5)$ or $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ depending on whether K is time-like or null, respectively. These two cases specify the most general supersymmetric solutions. Note that these are local conditions and that the case where this is a global statement has not yet been described.

A G-structure is the reduction of the $Spin(10,1)$ frame bundle to a principal G-sub-bundle. This structure can be specified by G-invariant tensors and/or spinors (such as (K, Ω, Σ) in our case). Since G defines a metric, one can take the covariant derivative of these G-invariant forms with respect to the Levi-Civita connection and classify the result into irreducible G-modules. Essentially, these modules measure the intrinsic torsion of the connection, or in physical terms, the modification of the connection due to fluxes. When all the modules are present, one has the most general type of G-structure, such as $SU(5)$ for time-like K for example. When all the modules vanish then this gives rise to special holonomy manifolds since, in physics language, there are no fluxes to modify the geometry.

Although we will not go into any more depth on the question of G-modules and that formalism, we note that one of the most useful consequences of this approach is that the local form of the geometry is almost completely determined by the G-structures. Along with the differential and algebraic relations described above, integrability arguments show that many of the components of the four-form field strength and metric of the corresponding solutions can be determined. Furthermore, this process has been refined to include the cases where more than one Killing spinor is preserved by the background [45].

This approach, coupled with the integrability of the Killing spinor equation, can be used to show that almost all of the components of the Einstein equations are satisfied. Assuming our geometry has Killing spinors and solves the equations of motion and the Bianchi identity, then, for the case of time-like K, the Einstein equations are automatically satisfied. The case of null K shows that one component of the Einstein equation needs to be imposed to obtain a full supersymmetric solution.

This has many practical benefits since the differential equations for the bilinear spinors (3.116) are first order, and the Einstein equations are second order and in general harder to solve. We shall exploit this fact in Chapter 3 when we re-derive the $\mathcal{N} = 1$ supergravity solution we examine in this thesis using this approach.

2.4 M5-branes wrapped on holomorphic 2-cycles in $\mathbf{C}^2$

In this section we briefly summarise the eleven-dimensional supergravity solution of fully localised M5-brane intersections [46–48]. Viewed from an M-theory perspective, this corresponds to an M5-brane with worldvolume $\mathbf{R}^{(1,3)} \times \Sigma$, where Σ is a Riemann surface of two complex dimensions. This is a holomorphic embedding which preserves $\mathcal{N} = 2$ (in $d = 4$) supersymmetry. This brane configuration is related, in the appropriate near-horizon limit, to $\mathcal{N} = 2$ supersymmetric gauge theories by the AdS/CFT correspondence [12].

2.4.1 The M5-brane configuration

This M-theory picture of an M5-brane wrapped on a holomorphic cycle of Σ has a ten-dimensional, Type IIA string theory interpretation. It describes a large class of Hanany-Witten [49] constructions. Generically, the Hanany-Witten setup involves D4-branes with worldvolume directions 01236 ending on NS5-branes extended in the 012345 directions. All the branes are located at $x^8 = x^9 = x^{10} = 0$. We can define the complex coordinates $v = x^4 + ix^5$ and $s = x^6 + ix^7$, where x^7 is the eleventh dimension (a circle of radius R). This complex structure plays an important part in defining the complex manifold Σ which the M5-brane wraps in the M-theory picture.

This Riemann surface Σ is in fact the Seiberg-Witten curve for the gauge theory [50]. The Seiberg-Witten differential also has an M-theory derivation [51] (see [52, 53] for a comprehensive review of these constructions). The BPS states correspond to minimal M2-branes whose boundary is on the M5-brane. The mass of the M2-brane gives the mass of the corresponding BPS-saturated state.

2.4.2 The supergravity solution

In the original approach, after solving the Killing spinor equations with these projection conditions and metric ansätze $\mathbf{R}^{(1,3)} \times Q^4 \times \mathbf{R}^{(3)}$, where Q^4 is a two-complex dimensional Calabi-Yau manifold, the following metric and field strength [46, 47] were discovered:

$$ds^2 = H^{-1/3} dx^2{}_{(1,3)} + 2H^{-1/3} g_{M\bar{N}} dz^M dz^{\bar{N}} + H^{2/3} dx^2{}_{(3)} \tag{2.27}$$

$$\begin{aligned} F_{M\bar{N}\alpha\beta} &= 2i\epsilon_{\alpha\beta\gamma}\partial_\gamma g_{M\bar{N}} \\ F_{M89(10)} &= -i\partial_M H \\ F_{\bar{N}89(10)} &= i\partial_{\bar{N}} H \\ g &= (g_{v\bar{v}} g_{s\bar{s}} - g_{s\bar{v}} g_{v\bar{s}}) \,. \end{aligned} \tag{2.28}$$

We denote with lower case Greek letters α, β, γ the totally transverse directions $8, 9, 10$, and capital letters M, N for the complex co-ordinates v, s.

The metric $g_{M\bar{N}}$ is constrained to be Kähler, with (square root) determinant g, and $H = 4g$ from asymptotic conditions. This is similar to what we would expect from the harmonic function rules [54] of orthogonally intersecting branes, but with extra off-diagonal components. These components are what effectively describe the brane in the relative transverse directions.

The equation of motion for F with a magnetic source J is

$$dF = J = J_{M\bar{N}} dz^M \wedge dz^{\bar{N}} \wedge dx^8 \wedge dx^9 \wedge dx^{10}, \tag{2.29}$$

where

$$J_{M\bar{N}} = -4i\,(\pi l_P)^3\,(\partial_M f)\,\left(\overline{\partial_N f}\right)\delta^2(f)\delta^3(r) \tag{2.30}$$

since the M5-brane is wrapped on a Riemann surface Σ, defined by a holomorphic function $f(v, s) = 0$ at $r = 0$, where r denotes the radial co-ordinate for the totally transverse space $\mathbf{R}^{(3)}$. This results, in terms of the Kähler potential for $g_{M\bar{N}}$, K, in the equation

$$8g(K) + \partial_\gamma \partial_\gamma K = -4\,(\pi l_P)^3\,|f|^2\,\delta^2(f)\delta^3(r) \tag{2.31}$$

which is related to the Monge-Ampère equation.

Taking the near-horizon limit

Once the brane construction of a particular gauge theory is known, one can try to describe the supergravity dual of the field theory. In the same spirit as the AdS/CFT correspondence, we identify the field theory parameters which should be kept fixed while taking a limit to decouple gravity and string modes.

Since we are interested in describing the gravity dual, we only need to solve these equations in the near-horizon limit. In this limit, we keep the gauge couplings and masses fixed while taking $l_P \to 0$. Looking at the example of a Hanany-Witten type IIA setup examined in [46], we have, for example, magnetically charged states represented by D2-branes stretched between the D4-branes and NS5-branes. Classically, they would have a mass

$$m = \frac{|v|L}{g_s(\alpha')^{3/2}} = \frac{|w|}{g_{YM}^2},$$

where $|v|$ is the coordinate distance between two D4-branes, and L is the distance between two NS5-branes. Thus, in the limit where we keep $w = v/\alpha'$ and the Yang-Mills coupling constant g_{YM} fixed, while taking $\alpha' \to 0$, the field theory states have finite mass.

Concretely, it was found that the relevant scalings of the supergravity variables in M-theory units, by defining w, t and y as follows, are:

$$\begin{aligned} w &= \frac{v}{{l_s}^2} = \frac{vR}{{l_P}^3} \\ t^2 &= \frac{r}{g_s {l_s}^3} = \frac{r}{{l_P}^3} \\ y &= \frac{s}{R}. \end{aligned}$$

Our expectations from the AdS/CFT duality suggests, for a conformal theory in a Hanany-Witten setup, a solution of the form of a warped product of AdS_5 with a non-compact six-dimensional manifold M_6. Requiring that the metric (2.27) can be written in this form places several constraints on the components of the Kähler metric $g_{M\bar{N}}$ which are not obviously related to the equations of motion. However, they are compatible and a solution has been found [48].

More general $\mathcal{N} = 2$ supersymmetric theories

Looking at the example of a conformal theory with two NS5-branes separated by $\frac{1}{g^2{}_{YM}}$ in the y-plane intersected (for gauge group $SU(N)$) by N infinite D4-branes, the holomorphic function $f(w, y)$ which describes this geometry, lifted to 11d, is

$$f = \left(y - \frac{1}{2g^2{}_{YM}}\right)\left(y + \frac{1}{2g^2{}_{YM}}\right) w^N.$$

This generalises for an arbitrary Riemann surface Σ (an arbitrary holomorphic function $f(w, y)$). In this case, the supergravity solution can be determined from the Kähler potential K which is given by [48]

$$\begin{aligned} K &= \frac{\pi N}{2t^2} \ln\left(\frac{\sqrt{t^4+|F|^4}+t^2}{\sqrt{t^4+|F|^4}-t^2}\right) + \frac{1}{2}|G|^2 \\ F^2 &= f^{1/N}, \end{aligned} \tag{2.32}$$

where in general N is defined as the degree of f as a polynomial in w. To find explicit solutions we need to solve

$$\left(\partial_y F^2\right)\left(\partial_w G\right) - \left(\partial_w F^2\right)\left(\partial_y G\right) = 1 \tag{2.33}$$

to find G. Whether this is easy or not depends on f.

Geometrically, the variables (F^2, G) can be thought of as local co-ordinates transverse and parallel to the M5-brane. The above equation is simply the statement that the Jacobian of the holomorphic co-ordinate transformation from (w, y) to (F^2, G) is equal to one. It is also the necessary condition for the metric

$$g_{M\bar{N}} \equiv 2\left(\partial_M F^2\right)\left(\overline{\partial_N F^2}\right) g + 1/2\left(\partial_M G\right)\left(\overline{\partial_N G}\right) \tag{2.34}$$

to have determinant g. The source equations (2.30), (2.31) reduce to the condition that g is a harmonic function in the five-dimensional transverse space with radial co-ordinate

$$\tilde{r} \equiv \sqrt{t^4+|F|^4}$$

so that $g = \frac{\pi N}{8\tilde{r}^3}$. These new co-ordinates appear to be naturally suited to describe this M5-brane configuration.

Calibrated surfaces

Defining the hermitian two-form $\omega_G = iG_{M\bar{N}} dz^M \wedge dz^{\bar{N}}$ (where we have rescaled the metric $G_{M\bar{N}} = H^{-1/3} g_{M\bar{N}}$), the spacetime metric is found to satisfy a (warped) Kähler calibration constraint

$$d_{\mathbf{C}^2}\left[H^{1/3}\omega_G\right] = 0, \tag{2.35}$$

with the derivative understood to be acting on the complex submanifold. This constraint is something which can be seen by generalised calibration arguments or otherwise [47, 55].

The non-vanishing components of the four-form field strength are:

$$\begin{aligned} F_{M\bar{N}\alpha\beta} &= 2i\epsilon_{\alpha\beta\gamma}\partial_\gamma g_{M\bar{N}} \\ F_{M89(10)} &= -i\partial_M H \\ F_{\bar{N}89(10)} &= i\partial_{\bar{N}} H \end{aligned} \tag{2.36}$$

These can be calculated simply once we note that the calibrating form Φ of the M5-brane is equal to its volume form, since it is a supersymmetric object. From the metric, this is can be seen to be

$$\begin{aligned} \Phi &= -iH^{-2/3} G_{M\bar{N}} dt \wedge dX^1 \wedge dX^2 \wedge dX^3 \wedge dz^M \wedge dz^{\bar{N}} \\ &= dV_{0123} \wedge \omega_G. \end{aligned} \tag{2.37}$$

Taking the Hodge dual of the exterior derivative and then using the constraint (2.35) gives the result (2.36), since $F_4 = *F_7 = *d\Phi$.

2.5 M5-branes wrapped on holomorphic 2-cycles in $\mathbf{C}^3$

For M5-branes wrapped on holomorphic 2-cycles in $\mathbf{C}^3$, one can generalise the previous construction, which describes the eleven-dimensional supergravity dual of $\mathcal{N} = 2$ field theories as the near-horizon limit of an M5-brane wrapped on a Riemann surface Σ, to the $\mathcal{N} = 1$ case. In particular, the eleven dimensional supergravity dual of certain $\mathcal{N} = 1$ field theories (so-called MQCD theories [14, 56]) is given by the near-horizon limit of an M5-brane wrapped on a three complex-dimensional Riemann surface Σ. MQCD is then the quantum field theory living on the 0123 part of an M5-brane with worldvolume $\mathbf{R}^{(1,3)} \times \Sigma$.

The idea of examining $\mathcal{N}=1$ theories by means of wrapping branes on Calabi-Yau manifolds has been extended to include the cases of generalised Calabi-Yau manifolds [57,58]. These stem from considering the extension to generalised complex geometry [59, 60]. These approaches have recently proved popular when dealing with flux compactifications of Type II theories to four dimensions, but we shall not examine them further here.

2.5.1 The M5-brane configuration

The idea is very similar to the $\mathcal{N}=2$ case, where we begin with a system of NS5-branes and D4-branes in Type IIA string theory. As an illustration, we can look at the simplest case of pure Yang-Mills with no matter. This is realised by two NS5-branes, denoted by $\mathrm{NS5}_1$ and $\mathrm{NS5}_2$. The $\mathrm{NS5}_1$ brane has worldvolume directions 012345, while the $\mathrm{NS5}_2$ brane has worldvolume directions 012389. They are separated in the 6 direction with the $\mathrm{NS5}_1$ brane defined to be on the left. We can then consider the inclusion of n D4-branes of finite extent in the 6 direction which are suspended between the NS5-branes. This configuration will then describe an $\mathcal{N}=1$ four-dimensional $\mathrm{SU}(n)$ field theory on the world-volume of the n finite D4-branes.

These configurations can be lifted to M-theory where they become an M5-brane wrapped on a non-compact Riemann surface Σ embedded in $\mathbf{C}^3$, generalising the $\mathcal{N}=2$ case of $\Sigma \subset \mathbf{C}^2$.

This is also equivalent to starting with the $\mathcal{N}=2$ configuration of Section 2.4 and rotating one of the NS5-branes from the 45 plane onto the 89 plane. This corresponds to turning on a mass for the adjoint scalar in the $\mathcal{N}=2$ vector multiplet, breaking the supersymmetry to $\mathcal{N}=1$. More general setups describing field theories with different gauge groups and matter have been constructed (see for instance [61] and related papers). For a similar analysis from the Type IIB viewpoint see also [62,63].

2.5.2 The supergravity solution

The supersymmetry preserving solutions of eleven-dimensional supergravity relevant for describing the M5-brane setup were described in [31]. The method is very similar to the $\mathcal{N} = 2$ case, so we shall go straight to the results. The solution was found to be:

$$\begin{aligned} ds^2 &= H^{-1/3}dx^2{}_{(1,3)} + 2H^{1/6}g_{M\bar{N}}dz^M dz^{\bar{N}} + H^{2/3}dy^2 \qquad (2.38) \\ \det g &= H \\ F &= \partial_y(\omega \wedge \omega) - i\partial(H^{1/2}\omega) \wedge dy + i\bar{\partial}(H^{1/2}\omega) \wedge dy \qquad (2.39) \\ \bar{\partial}(\omega \wedge \omega) &= 0 = \partial(\omega \wedge \omega). \qquad (2.40) \end{aligned}$$

In the above equations, the z^M are holomorphic coordinates:

$$\begin{aligned} z^1 &= v = x^4 + ix^5 \\ z^2 &= w = x^6 + ix^7 \\ z^3 &= s = x^8 + ix^9. \end{aligned}$$

The metric (2.38) is of the form $\mathbf{R}^{(1,3)} \times M_6 \times \mathbf{R}^{(1)}$, where M_6 is a Calabi-Yau 3-fold and $y = x^{(10)}$ denotes the remaining totally transverse direction. Also, ∂ denotes the $(1,0)$ exterior derivative $\partial = dz^M \partial_M$ in $\mathbf{C}^3$. The metric tensor $g_{M\bar{N}}$ is Hermitian, a property we shall use in the following calculations. It has an associated hermitian 2-form

$$\omega = ig_{M\bar{N}}dz^M \wedge dz^{\bar{N}} \qquad (2.41)$$

which is useful in expressing the field strength F in a more elegant form. One can check that the $\mathcal{N} = 2$ solution satisfies the above constraints.

Calibrated surfaces

These non-vanishing components of F can again be worked out easily from noticing that the calibrating form Φ of the M5-brane is equal to its volume form, since it is a supersymmetric object. From the metric, this is can be seen to be

$$\begin{aligned}\Phi &= iH^{-1/2}g_{M\bar{N}}dt\wedge dX^1\wedge dX^2\wedge dX^3\wedge dz^M\wedge dz^{\bar{N}}\\ &= dV_{0123}\wedge\omega. \end{aligned}\tag{2.42}$$

Taking the Hodge dual of the exterior derivative of this form gives the result (2.39), since $F_4 = *F_7 = *d\Phi$.

The spacetime metric is found to satisfy a co-Kähler calibration constraint

$$d_{\mathbf{C}^3}(\omega\wedge\omega) = 0 \;=\; d_{\mathbf{C}^3} * \omega, \tag{2.43}$$

where the exterior derivative and Hodge duality operation naturally take place in the $\mathbf{C}^3$ submanifold. This can be seen from generalised calibration arguments [31, 64], for example.

2.6 The M5-brane worldvolume action

Since we are interested in looking at the worldvolume gauge theory of probes in our wrapped M5-brane backgrounds, we briefly review the M5-brane worldvolume action. Historically, it was one of the last to be fully understood because of the inherent difficulty in dealing with self-dual forms in a covariant way.

The effective theory describing small fluctuations of the M5-brane should be a six-dimensional worldvolume theory with sixteen supersymmetries. This should include five scalars, a self-dual two-form field and eight fermions. The scalars and fermions are associated with fluctuations of the brane in target superspace along directions that are transversal to the brane worldvolume. The self-dual field, which carries three physical degrees of freedom, intrinsically propagates in the worldvolume. This chiral field was the main stumbling block in the construction of a Lagrangian description of the M5-brane action.

The question of how to both construct a gauge-invariant action, and one which reproduces the self-duality condition as an equation of motion, was solved by making use of an auxiliary scalar field. The construction of a non-covariant form of the M5-brane action was shown to be equivalent provided the scalar was chosen suitably. Finally, the action was coupled to a bosonic background of eleven-dimensional supergravity. A novel feature that was observed is that the local worldvolume supersymmetries of the action responsible for the self-duality properties require the presence of the Wess-Zumino term for the eleven-dimensional gauge field coupling to be consistent. In other types of branes, this was always a consequence of κ-symmetry.

The Action

The dynamics of the M5-brane probe are determined by its worldvolume action, the so called PST action [65] (see also [66]). In the PST formalism the worldvolume fields are a self-dual three-form field strength $H = dB_2$ and an auxiliary scalar field a (the PST scalar). The (bosonic) action is the sum of three terms:

$$S = \tau_5 \int d^6\sigma \left[\mathcal{L}_{DBI} + \mathcal{L}_{\mathcal{H}\tilde{\mathcal{H}}} + \mathcal{L}_{WZ}\right] \tag{2.44}$$

where the tension of the M5-brane is denoted by τ_5. In the action (2.44) the worldvolume field strength H is combined with the pullback $P[C^{(3)}]$ of the background three-form potential $C^{(3)}$ to form the field $\mathcal{H}$:

$$\mathcal{H} = H - e^{-\phi} P[C^{(3)}].$$

We can also define the field $\tilde{\mathcal{H}}$ as follows:

$$\tilde{\mathcal{H}}^{mn} = \frac{1}{3!\sqrt{-\det G}} \frac{1}{\sqrt{-(\partial a)^2}} \epsilon^{mnlpqr} \partial_l a \mathcal{H}_{pqr} \tag{2.45}$$

with G being the induced metric on the M5-brane worldvolume (see Appendix A for the conventions used throughout this paper).

The explicit expressions for the three terms in the action are:

$$\mathcal{L}_{DBI} = -\sqrt{-\det\left(G_{mn} + i\tilde{\mathcal{H}}_{mn}\right)} \tag{2.46}$$

$$\mathcal{L}_{\mathcal{H}\tilde{\mathcal{H}}} = \frac{1}{24(\partial a)^2} \epsilon^{lmnpqr} \mathcal{H}_{pqr} \mathcal{H}_{mns} G^{st} \partial_l a \partial_t a \tag{2.47}$$

$$\mathcal{L}_{WZ} = \frac{1}{6!}\epsilon^{lmnpqr}\left[P[C^{(6)}]_{lmnpqr} + 10\mathcal{H}_{lmn}P[C^{(3)}]_{pqr}\right]. \tag{2.48}$$

As discussed in [65], the scalar field a is an auxiliary field, which, by fixing its gauge symmetry, can be eliminated from the action at the expense of losing manifest covariance. To achieve agreement with the non-covariant formulation of the M5-brane action, one can fix the gauge $a = \sigma^5$ such that $\partial_{\hat{\mu}} a = \delta^5_{\hat{\mu}}$ and $B_{\mu 5} = 0$, which is allowed by the gauge symmetries of the PST action.

2.7 Type IIA Hanany-Witten models and gauge theory

As mentioned, the M-theory picture of an M5-brane wrapped on a 2-cycle has a ten-dimensional string theory interpretation in terms of intersecting brane configurations. These in turn describe large classes of supersymmetric gauge theories. This relation and the appropriate four-dimensional gauge theory was discovered by Witten [35], building on earlier work with Hanany [49]. These provided an excellent geometrical description of the important analysis that had been done earlier on the exact low-energy effective action of $\mathcal{N} = 2$ gauge theories [50]. Many features of the gauge theory such as the the gauge group, the running of the coupling and the matter content can thus be understood in simple geometrical terms.

Basic Construction

Consider Type IIA string theory on $\mathbf{R}^{(1,9)}$ with co-ordinates $x^0, \ldots, x^6, x^8 x^9, x^{(10)}$ (in our notation x^7 is the eleventh dimension $\{$a circle of radius $R\}$) . Generically, the simplest Hanany-Witten setup involves N_c "colour" D4-branes with worldvolume directions 01236 ending on a pair of NS5-branes extended in the 012345 directions. If we take the D4-branes to be stretched and ending on the NS5-branes, then the low energy effective theory on these D4-branes is in fact an $SU(N_c)$ $(3+1)$-dimensional super Yang-Mills theory with 8 supercharges. The effective gauge coupling of this theory would be

$$\frac{1}{g_{YM}^2} = \frac{L_6}{g_s l_s} \tag{2.49}$$

where we have denoted the separation of the NS5-branes by L_6, and g_s, l_s are the string coupling constant and the string length.

Now, in order to decouple the degrees of freedom of the bulk from the colour branes we take the double scaling limit $g_s \to 0$, $L_6/l_s \to 0$, $g_{YM} = constant$. The fact that one can take this double scaling limit is non-trivial and ultimately resolved by analysing the Little String Theory living on the worldvolume of the NS5-brane. This turns out to be described by an exactly solvable CFT and an analysis of branes and strings in this holographic description confirms the heuristic arguments regarding these strings and branes in Type IIA.

The light perturbative degrees of freedom of the gauge theory correspond to strings which are stretched between the N_c D4-branes, yielding an $\mathcal{N} = 2$ vector multiplet transforming in the adjoint representation of $SU(N_c)$. The distance between a pair of D4-branes correspond to the vacuum expectation values of the adjoint scalars in the vector multiplet and so parametrise the Coulomb branch.

One may also add hypermultiplets in the fundamental representation in the gauge theory by adding semi-infinite D4-branes to this construction. They can either end on one NS5-brane and extend to $x^6 = \infty$ (N_R) or end on the other and extend to $x^6 = -\infty$ (N_L). These are equivalent with respect to the gauge theory but it is only when we choose all $N_f = N_L + N_R$ semi-infinite D4-branes to be of the same type that the global $SU(N_f)$ flavour symmetry is evident from the brane configuration. The hypermultiplets transforming in the fundamental of $SU(N_c)$ are realised by open strings connecting the N_c and the N_f D4-branes.

An alternative way of adding matter transforming in the fundamental representation can be achieved in this setup by adding N_f "flavour" D4-branes stretched between the NS5-brane and a D6-brane with worldvolume 012389(10). According to the s-rule, no two D4-branes may connect the same NS5-brane and D6-brane if supersymmetry is to be preserved. The hypermultiplets would be strings stretching between one "colour" D4-brane and one "flavour" D4-brane, so they transform in the fundamental representations of the $SU(N_c)$ gauge group and the global flavour symmetry group. In addition, BPS states such as monopoles and dyons are realised by D2-branes ending on the D4-NS5-brane closed cycle, with the topology of a disk. Furthermore, one can also include BPS vortices by adding D2-branes in certain ways which we will have occasion to examine later.

If we define the complex co-ordinate $v = x^4 + ix^5$, then we can see that the $U(1)_R \times SU(2)_R$ R-symmetry of the classical $\mathcal{N} = 2$ theory corresponds to the rotational symmetry of our brane construction. The $U(1)_R$ symmetry is given by the rotational symmetry of the v-plane, while the $SO(3)$ rotational symmetry of the 89(10) space gives the $SU(2)_R$.

Another interesting feature of this brane construction is the correct description of the renormalisation group flow of the coupling constant. Essentially, the ends of the D4-brane are co-dimension two objects (vortices) in the NS5-brane worldvolume which induce a logarithmic (asymptotically) bending of the NS5-brane in the x^6 direction. Heuristically, the D4-branes are pulling on the NS5-branes, distorting their worldvolume. The N_c D4-branes pull in whereas the $N_f = N_L + N_R$ D4-branes pull out. Therefore, in the Hanany-Witten construction we are considering, this logarithmic bending is given by

$$\frac{1}{g_{YM}^2} = \frac{L_6(v)}{g_s l_s} \sim (2N_c - N_f) \ln |v|. \tag{2.50}$$

We note that the coefficient is exactly that of the one-loop (perturbatively exact) beta-function calculation for $\mathcal{N} = 2$ gauge theories, provided that $|v|$ is interpreted as an energy scale. Since this brane configuration restricts the motion supersymmetric brane probes in other directions, one may also take $|v|$ as the holographic co-ordinate (radial co-ordinate) in a UV/IR correspondence in analogy with AdS/CFT.

The effective theta angle is determined by the separation in the x^7 direction between the $\alpha - 1^{th}$ and α^{th} NS5-branes [35]. Essentially, the NS5-branes are generically at different points in the M-theory direction, and changing this difference in phase changes the effective theory and corresponding theta angle. If we set

$$\tau_\alpha = \frac{\theta_\alpha}{2\pi} + \frac{4\pi i}{g_{YM\alpha}^2}, \tag{2.51}$$

which actually corresponds to the complex coupling of the $\mathcal{N} = 2$ gauge theory, then in terms of $y = (x^6 + ix^7)/R$ (with distances measured in M-theory units), we have

$$-i\tau_\alpha(v) = y_\alpha(v) - y_{\alpha-1}(v) \qquad (2.52)$$

The coefficient of the right hand side has been set to one by requiring that under $x^7_\alpha \to x^7_\alpha + 2\pi R$, the theta angle changes by $\pm 2\pi$.

There are also quantum effects that can be effectively described by these constructions such as monopole and instanton contributions. These BPS states have geometrical interpretations in terms of branes. Semi-classical BPS states in the gauge theory such as magnetic monopoles and dyons are realised by D2-branes with the topology of a disk that are bounded by a D4-NS5-D4-NS5 cycle. In the near horizon limit, the mass of the monopole is proportional to the area of the D2-brane, which we recall is $\frac{|w|}{g^2_{YM}}$, which agrees with field theory computations. We also note that these states become massless when all the D4-branes are coincident, that is, at the root of the Coulomb branch.

One may also include instantons in the form of Euclidean D0-branes which coincide with the "colour" D4-branes extended in the x^6 direction, lying between two NS5-branes. These are possible because the D4-brane worldvolume action contains a Wess-Zumino term that includes a coupling to a one-form A_1 RR gauge potential. The D0-branes will then carry an instanton charge that is proportional to the electric coupling to this gauge field. Furthermore, since these are 1/2-BPS states, these instanton corrections will give rise to a repulsive force between "colour" D4-branes and prevent them from coinciding. Additionally, the D0-brane instanton will contribute non-perturbative corrections to the mass of the D2-brane monopole of order Λ.

There are further refinements and modifications that can be done to this setup, including adding BPS vortices, for example, and we shall examine those in later sections. However, it should be clear that many details of the gauge theory can be understood in geometric terms. An effective way of calculating such field theory parameters is by using branes to probe this geometry. Previous flat space analysis can be compared with probing the supergravity background which includes the backreaction of the brane construction. This is the subject of much of the work in the next chapter.

Lift to M-theory

We can go further and consider the lift to M-theory, since then the singular intersections of the D4-branes and NS5-branes will be smoothed out. In eleven dimensions, the requirement that the resulting configuration preserves the right degree of supersymmetry places constraints on the shape of the branes. These have been solved in an elegant fashion and we proceed to describe the results.

In our lift to eleven dimensions, we add the direction x^7, which as we mentioned before, shall be a circle of radius R. Using the relationship between D4-branes and NS5-branes in ten dimensions and M5-branes in eleven, we can easily see that the Hanany-Witten construction becomes some sort of M5-brane configuration. In fact, the D4-branes are M5-branes that wrap the x^7 direction, whereas the NS5-branes are M5-branes that do not. We can immediately deduce that for $R \neq 0$ the D4-brane will spread out in this direction, and, in particular, the boundary would not be contained within an NS5-brane worldvolume. Therefore, we must conclude that the configuration must be deformed in a particular manner which still retains supersymmetry.

It is useful to define the single valued complex co-ordinate $t = \exp(-s/R)$, where $s = x^6 + ix^7$. From (2.13), the M5-brane must be embedded holomorphically with respect to v, t. What actually happens then is that the collection of D4-branes and NS5-branes can be described by a single M5-brane with worldvolume $\mathbf{R}^{(1,3)} \times \Sigma$. So the embedding is described by a Riemann surface Σ which can be constructed as the vanishing locus of a polynomial $F(v,t)$. The degree in v of this polynomial is N_c and corresponds to the number of "colour" branes in the Hanany-Witten model. We also expect $F(v,t)$ to be quadratic in t since we consider a model with only two NS5-branes for simplicity. Likewise, semi-infinite D4-branes at a position $v = m_i$ correspond to asymptotic regions of the surface where $v \to m_i$ and $t \to 0$. If we denote the classical positions of the N_c D4-branes by ϕ^a, then the curve $F(v,t)$ can be shown to be specified by [35]

$$t^2 + t\prod_{a=1}^{N_c}(v-\phi^a) + \Lambda^{2N_c-N_f}\prod_{i=1}^{N_f}(v-m_i) = 0 \tag{2.53}$$

where Λ is a constant that can be identified with the dynamically generated QCD scale of the theory.

We note that the theta angle is encoded in this curve. This can be seen by rescaling t in such a way that the curve becomes

$$\tilde{t}^2 = \frac{B(v)^2}{4} - \Lambda^{2N_c-N_f}\prod_{i=1}^{N_f}(v-m_i). \tag{2.54}$$

Then we see that $F(v,t)$ is invariant under a phase shift $\tilde{t} \to e^{-i\theta}\tilde{t}$, which corresponds to movement along the x^7 direction, $x^7 \to x^7 + \theta$. Thus, once the M-theory circle is of finite radius, there is a new direction in which the NS5-branes can move, and this corresponds to the theta angle of the theory.

This Riemann surface Σ, specified by $F(v,t)$, is in fact the Seiberg-Witten curve for the gauge theory [50]. The classical geometry in eleven dimensions is then found to very elegantly describe all the instanton effects of the gauge theory. The fact that the D4-branes were thickened in the lift to M-theory can be interpreted as the instanton induced repulsion we mentioned previously, which corresponds to non-dynamical winding modes along the M-theory circle. The mass of the BPS monopole states, which correspond to minimal M2-branes whose boundary is on the M5-brane, are also modified because of this, in agreement with our earlier discussion. In fact, M2-branes of different topology describe all the matter in the gauge theory, from baryons to mesons. The Seiberg-Witten differential from which the masses of BPS states can be calculated also has an M-theory derivation [51] (see [52, 53] for a comprehensive review of these constructions).

So we have, indeed, gained a great deal by this lift to M-theory, but we also have some undesirable features to contend with. It contains, among other things, extra Kaluza-Klein states from the compact direction around which the M5-brane is wrapped. It is, in fact, a six-dimensional theory. These momentum modes are the reason why non-holomorphic quantities are not manageable in this limit. However, unbroken supersymmetries preserve holomorphic quantities such as masses of BPS states and superpotentials, which can be calculated exactly.

$\mathcal{N} = 1$ MQCD

One can further modify the above construction by rotating one of the NS5-branes in such a way that supersymmetry is softly broken to $\mathcal{N} = 1$ MQCD. The idea is to rotate one of the NS5-branes of the $\mathcal{N} = 2$ Hanany-Witten construction by an arbitrary angle in the (45,89) space. Performing this rotation breaks a further $\frac{1}{2}$ supersymmetry, resulting in only four supercharges.

This corresponds to turning on a mass for the adjoint scalar in the $\mathcal{N} = 2$ vector multiplet, breaking the supersymmetry to $\mathcal{N} = 1$. The reason is that after the rotation, the "colour" D4-branes are forced to either move toward the origin or attach themselves to flavour branes. Since this translation causes the D4-branes to stretch, the $\mathcal{N} = 1$ chiral multiplet containing these adjoint scalars acquires a mass via the superpotential term. This is commonly referred to as soft breaking. More general setups describing field theories with different gauge groups and matter have been constructed (see for instance [61] and related papers).

Now, breaking supersymmetry further will both help and hinder us in our quest to describe a QCD-like theory from M-theory. In fact, MQCD is said to be in the same universality class as $\mathcal{N} = 1$ QCD, and describes many qualitative features of $\mathcal{N} = 1$ QCD such as confinement [14], flux tubes, Seiberg duality and spontaneous breaking of discrete chiral symmetry [67]. We shall have the chance to discuss extended objects such as domain walls in this theory in later sections. However, the loss of some supersymmetry means that, for example, the Kähler potential is not protected from KK mode corrections.

When the quantum theory is studied using the lift to M-theory, the physics is again determined by a Riemann surface Σ, which is now embedded in the six-dimensional complex space spanned by v, t, s. We shall not be too interested in the exact form of the curve for this MQCD theory, but just to note that it is specified by two complex equations in these co-ordinates. Another way of saying this is that the M5-brane is wrapped on a holomorphic 2-cycle in $\mathbf{C}^3$.

We shall be interested in probing the supergravity background that these brane configurations generate with appropriate supersymmetric M5-brane probes in order to deduce relevant field theory parameters. Taking into account the backreactions of these branes on the background will also lead us to discuss M-theory structure groups and calibrations thereof. These setups provide a geometric illustration of structure groups.

2.8 Supersymmetric gauge theories: $\mathcal{N}=1$ and $\mathcal{N}=2$

In this section we present and discuss briefly the form of the $\mathcal{N}=1$ and $\mathcal{N}=2$ Lagrangians of supersymmetric Yang-Mills theories. Without going into too much detail, there are numerous reviews and books on this subject [68–72].

The $\mathcal{N}=1$ Lagrangian

We start with the $\mathcal{N}=1$ Lagrangian considering scalar and vector multiplets since they serve as building blocks for the construction of the $\mathcal{N}=2$ Lagrangian.

Our field theory will contain on-shell $\mathcal{N}=1$ scalar multiplets (ψ_α, ϕ), which consist of a Weyl fermion ψ_α and a complex scalar ϕ, as well as an on-shell vector multiplets (A_M, λ_α), which include the customary gauge field A_M and λ_α is the gaugino Majorana fermion.

In the superspace formalism, the scalar multiplet is represented by a chiral superfield Φ^i satisfying $\bar{D}_{\dot{\alpha}}\Phi = 0$, and the vector multiplet by a real superfield V satisfying $V = V^\dagger$.

In conventional superfield notation, the most general non-Abelian gauge kinetic and self-interaction term is built from the gauge field strength $W_\alpha = \frac{1}{8}\bar{D}^2 e^{2V} D_\alpha e^{-2V}$ and the chiral superfields Φ^i as follows

$$\mathcal{L}_G = \int d^2\theta \tau_{ab}(\Phi^i) W^a W^b + c.c. \tag{2.55}$$

Here a, b stand for the gauge index running over the adjoint representation of the gauge group and the functions $\tau_{ab}(\Phi^i)$ are required to be complex analytic. If we regard this as a constant chiral superfield, then it reduces to $\tau_{cl} = \theta/2\pi + 4\pi i/g_{YM}^2$. If we expand this in terms of the component fields we find, for the case of constant τ_{ab}, the result

$$\mathcal{L} = -\frac{1}{4g_{YM}^2} F_{MN}^A F^{AMN} + \frac{\theta_{YM}}{32\pi^2} F_{MN}^A \tilde{F}^{AMN} + fermions \tag{2.56}$$

which shows the usual Yang-Mills gauge kinetic and theta terms where $\tilde{F}$ denotes the dual field strength. We may also add a Kähler potential term $K(e^V \Phi^i, (\Phi^i)^\dagger)$ and a superpotential term $U(\Phi^i)$ to this Lagrangian via D- and F-terms.

The $\mathcal{N} = 2$ Lagrangian

This field theory contains a vector multiplet $(A_M, \lambda_\pm, \phi)$ which has the same field content as the sum of an $\mathcal{N} = 1$ scalar multiplet and an $\mathcal{N} = 1$ vector multiplet. It also contains hypermultiplets with component fields $(\psi_+, H_\pm, \psi_-)$ where $\psi_\pm$ form a Dirac spinor and $H_\pm$ are complex scalars. Under the $SU(2)_R$ symmetry the spinor fields are singlets whereas the scalars transform as a doublet.

To construct our $\mathcal{N} = 2$ Lagrangian we may use our results from (2.55). For our purposes, it is enough to note that this requires the scaling of the chiral superfield $\Phi \to \Phi/g_{YM}$ in (2.55). Using the conventional $\mathcal{N} = 2$ superspace formalism we may express the fields in terms of the $\mathcal{N} = 2$ chiral superfield Ψ which is composed of $\mathcal{N} = 1$ superfields in an appropriate manner.

One can verify that in terms of Ψ, the general $\mathcal{N} = 2$ Lagrangian for gauge fields is given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi} \mathrm{Im}\ \mathrm{Tr} \int d^2\theta d^2\tilde{\theta} \mathcal{F}(\Psi) \\ &= \frac{1}{8\pi} \mathrm{Im} \left(\int d^2\theta\ \mathcal{F}_{ab}(\Phi) W^{a\alpha} W_\alpha^b + 2 \int d^2\theta d^2\bar{\theta} (\Phi^\dagger e^{2g_{YM}V})^a \mathcal{F}_a(\Phi) \right). \end{aligned} \tag{2.57}$$

The second line expresses the same Lagrangian in terms of $\mathcal{N} = 1$ superfields. Here $\mathcal{F}_a(\Phi) = \partial\mathcal{F}/\partial\Phi^a$, $\mathcal{F}_{ab}(\Phi) = \partial^2\mathcal{F}/\partial\Phi^a\partial\Phi^b$ and $\mathcal{F}$ is referred to as the $\mathcal{N} = 2$ prepotential. The exact determination of this function is the subject of the Seiberg-Witten analysis.

It is useful to consider the Wilsonian effective action which is obtained by completely integrating out all massive states as well as integrating out all massive excitations above a fixed scale. For gauge group $\mathcal{G}$ only $n = \text{rank}(\mathcal{G})$ massless $\mathcal{N}=2$ Abelian $U(1)$ gauge supermultiplets will remain[1]. Thus the Wilsonian action will describe n massless $U(1)$ gauge supermultiplets, with fields $(A^j_M, \lambda^j_\pm, \phi^j)$, as a function of the vacuum moduli parameters a_j, which are the vevs of ϕ^j. For $\mathcal{N}=2$, this turns out to be given by

$$\mathcal{L} = \text{Im}(\tau_{ab})F^a_{MN}F^{bMN} + \text{Re}(\tau_{ab})F^a_{MN}\tilde{F}^{bMN} + \text{Im}(\partial_M\bar{\phi}^a\partial^M\phi_{Da}) + fermions \quad (2.58)$$

with

$$\phi_{Da} = \frac{\partial\mathcal{F}(\phi)}{\partial\phi^a} \qquad \tau_{ab} = \frac{\partial^2\mathcal{F}(\phi)}{\partial\phi^a\partial\phi^b}. \quad (2.59)$$

Here it is understood that the gauge scalar ϕ^j takes on the vev $a^j \in \mathbf{C}$ which are arbitrary complex numbers. Also, we note that $\mathcal{F}(a_j;m_a,\tau)$ is a holomorphic function of the coupling τ and the hypermultiplet masses m_a.

In terms of $\mathcal{F}(\phi)$, the Kähler potential is given by $K = \text{Im}(\phi^{\dagger a}\mathcal{F}_a(\phi))$. In terms of the vev a_b, the metric on the space of fields, and therefore, the metric on the space of Higgs vacua, is given by

$$ds^2 = g_{a\bar{b}}da^a d\bar{a}^b = \text{Im}\frac{d^2\mathcal{F}}{\partial a_a\partial\bar{a}_b}da^a d\bar{a}^b. \quad (2.60)$$

We shall re-derive this Kähler metric for the scalar kinetic term from a brane probe analysis in the next chapter taking into account the full geometry of the background. Previous flat space analysis was done in [73].

[1]Note that for special values of the vacuum moduli, one or more of the dyons that have been integrated out may also become massless. This phenomenon and the curves of marginal stability that describe it are central features of the Seiberg-Witten theory.

2.9 Summary

In this introductory chapter, we have discussed the ideas which are relevant for the results in this thesis. We began with an introduction to the condition for preservation of supersymmetry in eleven-dimensional supergravity by satisfying the Killing spinor equation. This led to a discussion of the projection conditions for supersymmetric branes, and their generalisation to branes wrapped on holomorphic curves.

This naturally led to discussing calibrations in flat backgrounds, and their relation to special holonomy manifolds. A brief summary of the most relevant ones was given. There followed a discussion on the bilinear spinor formalism we will be employing throughout, as well as a brief introduction to G-structures as a method of classifying supersymmetric supergravity solutions.

We summarised the supergravity solutions we shall be investigating in this thesis, as well as their near-horizon limit. This led to a short discussion on the related ten-dimensional Hanany-Witten models of which these solutions are the M-theory uplift. Some brief statements were made about their corresponding gauge theories. We included a small section on the M5-brane worldvolume and explained some of its more uncommon features. Finally, we discussed the main features of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ Yang-Mills Lagrangians, with a view of introducing terminology for the calculations of the next chapter.

Chapter 3

Brane probes in wrapped M5-brane backgrounds

3.1 M5-brane probe calculation

In this section we examine the induced action on an M5-brane which is probing the background of an M5-brane wrapped on a 2-cycle in $\mathbf{C}^2$. This corresponds to an $\mathcal{N} = 2$ Yang-Mills theory when considering the worldvolume action of the probe. In particular, we shall look at the complex scalar kinetic terms and verify that the metric of the space of fields is indeed Kähler, as we saw in the previous chapter. Since we have embedded the probe brane holomorphically with respect to the background, we will be able to calculate holomorphic quantities in the dual gauge theory. Since supersymmetry protects certain holomorphic quantities from perturbative g_s quantum corrections [61], classical computations involving the M5-brane can still determine these quantities exactly. This extends previous flat-space analysis [73, 74] to the case where the full geometry is taken into account.

3.1.1 Probe calculation of complex scalar moduli space

Our M5-brane probe will have a worldvolume of the form $\mathbf{R}^{(1,3)} \times \Lambda$ where Λ is a two-dimensional complex surface in Q^4 which is allowed to vary over $\mathbf{R}^{(1,3)}$. Also, we assign to it the worldvolume co-ordinates σ^m, z, $\bar{z}$ whose embeddings are holomorphic and of the form:

$$\begin{aligned} X^m &= \sigma^m \\ X^M &= X^M\left(z, \sigma^m, u_\alpha\left(\sigma^m\right)\right) \\ X^{\bar{N}} &= X^{\bar{N}}\left(\bar{z}, \sigma^m, u_{\bar{\beta}}\left(\sigma^m\right)\right) \\ X^a &= X^a\left(z, \bar{z}, \sigma^m, u_\alpha\left(\sigma^m\right), u_{\bar{\beta}}\left(\sigma^m\right)\right). \end{aligned}$$

For the purposes of this calculation we define $m = 0\ldots 3$, $M, N = F^2, G$ and X^a refers to the totally transverse directions $a = 8, 9, 10$ (the conventions are similar to those in [73] where a related calculation was performed). Also, z, $\bar{z}$ are arbitrary co-ordinates on the Riemann surface Λ which has u_α, $u_{\bar{\beta}}$ as its complex moduli.

We consider only small deviations from a supersymmetric embedding of the probe, so $\frac{\partial X^M}{\partial \sigma^m}$, $\frac{\partial X^a}{\partial z}$, $\frac{\partial X^a}{\partial \sigma^m}$, $\frac{\partial X^a}{\partial u_\alpha}$ and $\frac{\partial u_\alpha}{\partial \sigma^m}$ are small. This typically breaks all the supersymmetries, but since these are only very small deviations from the supersymmetric configuration we can expand the M5-brane probe action to quadratic order in these terms to find the metric on the moduli space.

As the five-brane action is invariant under worldvolume diffeomorphisms, we can always choose z and $\bar{z}$ in such a way that the induced metric on the Riemann surface is conformal, i.e. $g_{zz} = g_{\bar{z}\bar{z}} = 0$. As this will simplify things considerably, we will from now on assume this to be the case.

The first case we shall consider is an M5-brane probe with no worldvolume H field turned on and neglecting the WZ contribution of the action. We will also ignore the $z, \bar{z}$ dependence of the X^a. In this case the probe action reduces to

$$S = -\tau_5 \int d^6\sigma \sqrt{-\det\left(G_6\right)},$$

with G_6 the full six-dimensional worldvolume metric. Explicitly, the action induced from the background metric becomes:

$$S = \int d^4\sigma d^2 z 2H^{-1} g_{z\bar{z}} \sqrt{-\det\left(\eta_{mn} + L_{mn}\right)} \tag{3.1}$$

where

$$\begin{aligned} L_{mn} \;=\; & 2\left[\partial_m X^M \partial_n X^{\bar{N}}\left(g_{M\bar{N}} - g_{M\bar{z}}\frac{1}{g_{z\bar{z}}}g_{z\bar{N}}\right) + \partial_m u_\alpha \partial_n X^{\bar{N}}\left(g_{\alpha\bar{N}} - g_{\alpha\bar{z}}\frac{1}{g_{z\bar{z}}}g_{z\bar{N}}\right)\right. \\ & \left. + \partial_m X^M \partial_n u_{\bar{\beta}}\left(g_{M\bar{\beta}} - g_{M\bar{z}}\frac{1}{g_{z\bar{z}}}g_{z\bar{\beta}}\right) + \partial_m u_\alpha \partial_n u_{\bar{\beta}}\left(g_{\alpha\bar{\beta}} - g_{\alpha\bar{z}}\frac{1}{g_{z\bar{z}}}g_{z\bar{\beta}}\right)\right] \\ & + g\left[p_{\alpha\beta}\frac{\partial u_\alpha}{\partial\sigma^m}\frac{\partial u_\beta}{\partial\sigma^n} + p_{\alpha\bar{\beta}}\frac{\partial u_\alpha}{\partial\sigma^m}\frac{\partial u_{\bar{\beta}}}{\partial\sigma^n} + p_{\bar{\beta}\alpha}\frac{\partial u_{\bar{\beta}}}{\partial\sigma^m}\frac{\partial u_\alpha}{\partial\sigma^n}\right. \\ & + p_{\bar{\alpha}\bar{\beta}}\frac{\partial u_{\bar{\alpha}}}{\partial\sigma^m}\frac{\partial u_{\bar{\beta}}}{\partial\sigma^n} + p_{\alpha b}\frac{\partial u_\alpha}{\partial\sigma^m}\frac{\partial X^b}{\partial\sigma^n} + p_{\bar{\beta}b}\frac{\partial u_{\bar{\beta}}}{\partial\sigma^m}\frac{\partial X^b}{\partial\sigma^n} \\ & \left. + p_{a\alpha}\frac{\partial X^a}{\partial\sigma^m}\frac{\partial u_\alpha}{\partial\sigma^n} + p_{a\bar{\beta}}\frac{\partial X^a}{\partial\sigma^m}\frac{\partial u_{\bar{\beta}}}{\partial\sigma^n} + \delta_{ab}\frac{\partial X^a}{\partial\sigma^m}\frac{\partial X^b}{\partial\sigma^n}\right] \end{aligned} \tag{3.2}$$

and we have defined $g_{\alpha\bar{\beta}} \equiv \frac{\partial X^M}{\partial u_\alpha} g_{M\bar{N}} \frac{\partial X^{\bar{N}}}{\partial u_{\bar{\beta}}}$, $g_{M\bar{\beta}} \equiv g_{M\bar{N}}\frac{\partial X^{\bar{N}}}{\partial u_{\bar{\beta}}}$, $p_{\alpha\bar{\beta}} \equiv \frac{\partial X^a}{\partial u_\alpha}\delta_{ab}\frac{\partial X^b}{\partial u_{\bar{\beta}}}$ and $p_{a\bar{\beta}} \equiv \delta_{ab}\frac{\partial X^b}{\partial u_{\bar{\beta}}}$. In this notation, the spacetime metric $g_{M\bar{N}}$ is the same as that of Equation (2.34).

The second bracket which is multiplied by g only contributes terms which are at least cubic in small derivatives, so are typically higher order corrections and we will not analyse them here. The very last term is an exception since it remains quadratic but it nevertheless does not contribute to the effective four-dimensional theory we are interested in.

The reason is that, on the one hand, if the complex space Λ is non-compact, it would pick up an infinite mass term from the volume integral and could therefore be neglected in the four-dimensional field theory analysis. Whereas if the space Λ is compact, we can in principle perform an expansion in terms of Fourier modes, and the fact that the endpoints of our probe are by definition constrained in the X^a directions forces the zero modes to be at least linear in derivatives. These boundary conditions then imply that the last term will be of higher than quadratic order in derivatives, and therefore not contribute to our analysis.

Upon expansion of the action (3.1), the kinetic term for the scalars $X^M, X^{\bar{N}}$ and u_α reads

$$S_{kin} = \tau_5 \int d^4\sigma d^2 z \; g^{-1} g_{z\bar{z}} \frac{1}{2} Tr(L). \tag{3.3}$$

Looking at the quadratic terms in the complex moduli only, we find

$$S_{kin} = \tau_5 \int d^4\sigma \, \partial_m u_\alpha \partial^m u_{\bar{\beta}} K_{\alpha\bar{\beta}}, \tag{3.4}$$

where $K_{\alpha\bar{\beta}}$ is a Kähler metric given by

$$K_{\alpha\bar{\beta}} = \int_\Lambda d^2z \, g^{-1} \left(g_{\alpha\bar{\beta}} g_{z\bar{z}} - g_{\alpha\bar{z}} g_{z\bar{\beta}} \right). \tag{3.5}$$

This is Kähler up to total derivative boundary terms of the form

$$\int_\Lambda d^2z \left(\left(\partial_{\bar{z}} \bar{F}^2\right) \left(\partial_{\bar{\beta}} \bar{G}\right) - \left(\partial_{\bar{\beta}} \bar{F}^2\right) \left(\partial_{\bar{z}} \bar{G}\right) \right) \partial_z \left[\left(\partial_\alpha G\right) \left(\partial_\gamma F^2\right) - \left(\partial_\gamma G\right) \left(\partial_\alpha F^2\right) \right]$$

and

$$\int_\Lambda d^2z \left(\left(\partial_z F^2\right) \left(\partial_\alpha G\right) - \left(\partial_\alpha F^2\right) \left(\partial_z G\right) \right) \partial_{\bar{z}} \left[\left(\partial_{\bar{\gamma}} \bar{F}^2\right) \left(\partial_{\bar{\beta}} \bar{G}\right) - \left(\partial_{\bar{\beta}} \bar{F}^2\right) \left(\partial_{\bar{\gamma}} \bar{G}\right) \right],$$

where we have ignored contributions coming from the $g(\sum_{x,y} p_{xy})$ terms (the totally transverse fluctuations of the brane), as explained above. In this calculation we have also explicitly used the fact that the spacetime metric $g_{M\bar{N}}$ is Kähler.

These terms can be written as a total derivative straight away since they are a product of holomorphic and anti-holomorphic factors. In particular, if we define the one-forms

$$\Phi_\alpha = \left[\left(\partial_z F^2\right) \left(\partial_\alpha G\right) - \left(\partial_\alpha F^2\right) \left(\partial_z G\right) \right] dz, \tag{3.6}$$

then $d\Phi_\alpha = 0$ because of holomorphicity. So if we also define the scalars

$$B_{\alpha\gamma} = \left(\partial_\alpha G\right) \left(\partial_\gamma F^2\right) - \left(\partial_\gamma G\right) \left(\partial_\alpha F^2\right), \tag{3.7}$$

the boundary terms are of the form

$$\int_\Lambda dB_{\alpha\gamma} \wedge \bar{\Phi}_{\bar{\beta}} \text{ and } \int_\Lambda d\bar{B}_{\bar{\beta}\bar{\gamma}} \wedge \Phi_\alpha. \tag{3.8}$$

Evaluating them at the boundary results in

$$\int_{\partial\Lambda} B_{\alpha\gamma} \, \bar{\Phi}_{\bar{\beta}} \text{ and } \int_{\partial\Lambda} \bar{B}_{\bar{\beta}\bar{\gamma}} \, \Phi_\alpha. \tag{3.9}$$

So we can impose that these terms vanish at the boundary. For a non-compact probe, the asymptotic embedding is independent of the moduli, so clearly these boundary terms vanish (since $(\partial_\alpha F^2) = (\partial_\alpha G) = 0$ asymptotically), and hence the metric is Kähler. However, while a finite D4-brane probe in Type IIA can end on a background NS5-brane, this is only possible for a supersymmetric probe as an approximation for small $R > 0$, so for such a probe one should choose appropriate boundary conditions such that the boundary terms vanish.

Additionally, there are the mixed terms $\partial_\mu u_\alpha \partial^\mu X^{\bar{N}}$ and $\partial_\mu X^M \partial^\mu u_{\bar{\beta}}$ as well as the quadratic term of the complex scalars $\partial_\mu X^M \partial^\mu X^{\bar{N}}$. They all have Kähler metrics on their moduli space with boundary terms similar in form to the ones we have just analysed, giving the expected result that the moduli space of all the complex scalars is given by a Kähler metric.

From the expression for the Kähler metric (3.5) of the complex scalars with respect to the complex moduli, one can then obtain the standard form of the scalar kinetic terms of the $\mathcal{N} = 2$ effective Lagrangian in the usual ways (see for example [73]).

3.1.2 A simple example: the parallel brane probe

Another example is to probe the background with an M5-brane which is parallel to the background M5-brane configuration. This does not imply it is flat, but merely that it somehow reflects the shape of the background. We shall let our probe have worldvolume $0123z\bar{z}$, where $z = \sigma^4 + i\sigma^5$. This time we do not consider fluctuations in the complex moduli of the brane. We will let the probe have a time dependence on $Q^4 \times \mathbf{R}^3$. The embeddings are then

$$\begin{aligned} X^m &= \sigma^m \\ X^M &= X^M(z, \sigma^0) \\ X^{\bar{N}} &= X^{\bar{N}}(\bar{z}, \sigma^0) \\ X^a &= X^a(\sigma^0). \end{aligned}$$

The action for the kinetic scalar terms then becomes

$$S_{kin} = \tau_5 \int d^4\sigma d^2 z g^{-1} g_{z\bar{z}} \left(g_{0\bar{0}} - g_{0\bar{z}} \frac{1}{g_{z\bar{z}}} g_{z\bar{0}} + g \left(\partial_0 X^a \right)^2 \right) \tag{3.10}$$

where $g_{0\bar{0}} = \partial_0 X^M g_{M\bar{N}} \partial_0 X^{\bar{N}}$.

Now, for a probe which is parallel to the background we set $z = G$ which simplifies the above expression to

$$S_{kin} = \tau_5 \int d^4\sigma d^2 z \left(\left| \partial_0 F^2 \right|^2 + \frac{1}{2} \left(\partial_0 X^a \right)^2 \right). \tag{3.11}$$

This means that the brane sees a flat metric on the transverse directions which agrees with the expectation of a flat moduli space metric. Also, there is a trivial volume form which seems to suggest that these co-ordinates are a natural way to describe this configuration.

We shall return to more results from M5-brane probes shortly, but before we do that, we take a quick foray into an M2-brane probing the BPS spectra of the field theory.

3.2 M2-brane probe calculation

The main result of this section is to calculate the mass of BPS states in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. We shall be using an M2-brane as a probe of the supergravity background corresponding to completely localised M5-brane configurations in M-theory (or equivalently M5-branes wrapping 2-cycles in $\mathbf{C}^2$), which is the supergravity dual of a large class of such gauge theories. States corresponding to BPS monopoles are realised as two-branes ending on the five-branes. One example of this are the membranes in a Hanany-Witten type setup. In particular, we check whether this method provides corrections to the previous flat-space four-dimensional $\mathcal{N} = 2$ supersymmetric field theory analysis [74, 75].

3.2.1 Introductory remarks

There are two ways in which this can be done. In the probe analysis, we find a suitable complex structure in the hyper-Kähler part of the background in which to embed the M2-brane holomorphically, and then proceed to calculate the induced volume. We calculate the case of a static M2-brane and check it receives no corrections from the supergravity description.

The other method is based on the approach of calibrations [36, 40, 41, 76]. This relates the BPS bound to the central charge of the eleven-dimensional supergravity supersymmetry algebra. We take into account the generalisation of these calibration forms to include arbitrary background fields [77]. Again these topological charges give no corrections to the previous flat-space field theory calculations of the BPS monopole mass.

In the following, we shall establish the complex structure which the M2-brane probe should be holomorphically embedded with respect to, and proceed to calculate its worldvolume action. There follows a brief review of the concept of generalised calibrations and a calculation of the calibration bound for the M2-brane given the appropriate supersymmetric projection conditions. In both cases, we find no corrections to the previous flat-space field theory analysis.

3.2.2 M2-brane probe calculation

In this section, we will study the action of an M2-brane probe since it is known that minimal area membranes which end on M5-branes are related to the BPS states of $\mathcal{N} = 2$ gauge theories to which our supergravity background is dual. Our background is sourced by the M-theory configuration described in the last section, which has the topology $\mathbf{R}^{(1,3)} \times Q^4 \times \mathbf{R}^{(3)}$, up to warp factors, where Q^4 is a hyper-Kähler manifold.

Preliminaries

Consider an M2-brane probe with worldvolume $\mathbf{R} \times D$, where D, the spatial part of the M2-brane, is a two dimensional surface embedded in the manifold Q^4 given by our background. Apart from the warp factor, we know Q^4 is hyper-Kähler because Q^4 is actually a Calabi-Yau 2-fold, and all two-complex dimensional Calabi-Yau manifolds are automatically hyper-Kähler. This means that instead of the usual one complex structure, this geometry admits a family of inequivalent complex structures parametrised by a two-sphere S^2, with $SU(2)$ commutation relations between them. Also, in four dimensions, the hyper-Kähler condition implies Ricci flatness and should therefore admit a covariantly constant holomorphic two-form.

We denote by Σ the surface of the M5-brane which is embedded holomorphically in Q. Now, we wish to embed our M2 probe holomorphically so that its spatial part has a boundary $C = \partial D$ that lies on Σ, i.e. so the two-brane ends on the five-brane. To achieve this, the M2-brane will be embedded holomorphically with respect to some complex structure J' which is orthogonal to the complex structure J in which the M5-brane was embedded holomorphically. Given a complex structure J, the set of such J' for a hyper-Kähler manifold is parametrised by an S^1 that actually corresponds to the phase of the central charge of the BPS saturated state [74].

To further and completely distinguish between the different possibilities, we also require that the M2-brane probe satisfy the supersymmetry projection conditions.

Choosing the appropriate complex structure

For our particular background geometry, the five-brane is wrapped around the holomorphic curve Σ and the Killing spinors satisfy [78]:

$$\hat{\Gamma}_{0123}\hat{\Gamma}_{a\bar{b}}\epsilon = i\delta_{a\bar{b}}\epsilon \tag{3.12}$$

with a, b running over z^1, z^2. These projection conditions preserve 8 real components of ϵ and thus give $\mathcal{N} = 2$ supersymmetry in four dimensions.

Introducing the two-brane which ends on the five-brane requires the additional constraint [51]

$$\frac{1}{2}\epsilon^{\alpha\beta}\Gamma_0\Gamma_{I\bar{J}}\partial_\alpha W^I\partial_\beta W^{\bar{J}}\epsilon = \epsilon \tag{3.13}$$

where W^I now denotes the embedding of the two-brane with respect to a different complex structure.

Explicitly, if we rewrite the hyper-Kähler part of the metric in terms of the vielbeins

$$g_{M\bar{N}}dz^M dz^{\bar{N}} = |dZ^1|^2 + |dZ^2|^2 = e^a_M\left(\overline{e^b_N}\right)\delta_{a\bar{b}}dz^M dz^{\bar{N}},$$

the complex structure J, compatible with the M5-brane configuration, becomes

$$\begin{aligned} dZ^1 &= Re\left(e^1_M dz^M\right) + iIm\left(e^1_M dz^M\right) \\ dZ^2 &= Re\left(e^2_M dz^M\right) + iIm\left(e^2_M dz^M\right). \end{aligned} \tag{3.14}$$

We can now deduce the alternative complex structure J' that satisfies the projection conditions and the orthogonality constraint. In terms of the differentials, these are

$$\begin{aligned} dW^1 &= Re\left(e^1_M dz^M\right) + iRe\left(e^2_M dz^M\right) \\ dW^2 &= Im\left(e^1_M dz^M\right) - iIm\left(e^2_M dz^M\right). \end{aligned} \tag{3.15}$$

The M2-brane probe will be embedded holomorphically with respect to the co-ordinates W^1, W^2 in the above basis. As we shall see, we won't actually need to integrate the dW^1, dW^2 differentials, which simplifies the task considerably. Additionally, one can also trivially include an arbitrary phase which rotates the W^1, W^2 co-ordinates. We include this phase for completeness in the analysis of Section 3.2.3.

We can rewrite the M2-brane projection condition (3.13) in terms of the M5-brane holomorphic variables using Equation (3.15). In this language, the projection condition is

$$\left(\hat{\Gamma}_{0ab} + \hat{\Gamma}_{0\bar{a}\bar{b}}\right)\epsilon = \epsilon \tag{3.16}$$

with again $a, b = 1, 2$. This additional constraint cuts the number of supersymmetries by half (leaving four real supersymmetries), expressing the fact that the M2-brane is a BPS state in the worldvolume theory of the M5-brane.

Probe calculation

We shall now consider our background spacetime $\mathbf{R}^{(1,3)} \times Q^4 \times \mathbf{R}^{(3)}$ with metric

$$ds^2 = H^{-1/3}dx^2{}_{(1,3)} + 2H^{-1/3}g_{M\bar{N}}dz^M dz^{\bar{N}} + H^{2/3}dx^2{}_{(3)}, \tag{3.17}$$

where

$$g_{M\bar{N}} \equiv 2\left(\partial_M F^2\right)\left(\overline{\partial_N F^2}\right) g + 1/2\left(\partial_M G\right)\left(\overline{\partial_N G}\right) \tag{3.18}$$

and g is given by $g = \frac{\pi N}{8\bar{r}^3}$. We define the spacetime indices $m = 0, \ldots, 3$ and $M, N = F^2, G$ that run over the Lorentzian part and the hyper-Kähler part respectively.

The worldvolume co-ordinates of our M2-brane shall be $(t, \sigma, \bar{\sigma})$, where we have complexified the spatial part of the brane (with $\sigma = \sigma^1 + i\sigma^2$) for future convenience. These will have holomorphic embeddings of the form

$$\begin{aligned} X^0 &= t \\ W^I &= W^I(\sigma) \\ W^{\bar{J}} &= W^{\bar{J}}(\bar{\sigma}). \end{aligned} \tag{3.19}$$

This static probe will provide information about the mass of BPS states of the dual gauge theory. The action of an M2-brane is given by

$$S_{M2} = -\tau_2 \int d^3\sigma \sqrt{-\det\left(G_{ij}\right)} + \int \Xi \tag{3.20}$$

where τ_2 is the tension, G_{ij} is the pullback of the spacetime metric onto the two-brane and Ξ is the pullback of the spacetime three-form potential. Note that this last term vanishes for our particular embedding and so does not contribute in the analysis.

In terms of real co-ordinates, we can define $z^M = x^M + iy^M$ and split the complex vielbein into real and imaginary parts $e^a_M = \alpha^a_M + i\beta^a_M$. The holomorphy condition on the worldvolume induced co-ordinates,

$$\frac{\partial W^1}{\partial \bar{\sigma}} = \frac{\partial W^2}{\partial \bar{\sigma}} = 0$$

then gives a set of four constraints on the vielbeins. We can simplify these equations by defining

$$\begin{aligned} A^a_i &\equiv \alpha^a_M \partial_i x^M - \beta^a_M \partial_i y^M \\ B^a_i &\equiv \beta^a_M \partial_i x^M + \alpha^a_M \partial_i y^M. \end{aligned}$$

In terms of these new variables, our holomorphy constraints imply

$$A_1^1 = A_2^2,\ A_2^1 = -A_1^2,\ B_1^1 = -B_2^2,\ B_2^1 = B_1^2.$$

The induced metric can then be written

$$G_{ij} = \delta_{a\bar{b}}\left[A_i^a + iB_i^a\right]\left[A_j^b - iB_j^b\right] + \left[i \leftrightarrow j\right].$$

We can simplify this further by defining new complex variables

$$C_i^a = A_i^a + iB_i^a$$

which transforms the constraints to

$$C_1^1 = \overline{C_2^2} \text{ and } C_2^1 = -\overline{C_1^2}. \tag{3.21}$$

More concretely, in terms of our complex vielbeins, we have

$$C_i^a = e_M^a \frac{\partial z^M}{\partial \sigma^i}. \tag{3.22}$$

Finally, the induced metric can be written in the form

$$G_{ij} = \delta_{a\bar{b}} e_M^a e_{\bar{N}}^{\bar{b}} \frac{\partial z^M}{\partial \sigma^i}\frac{\partial z^{\bar{N}}}{\partial \sigma^j} + [i \leftrightarrow j] = \delta_{a\bar{b}} C_i^a \left(\overline{C_j^b}\right) + [i \leftrightarrow j] \tag{3.23}$$

which can be checked explicitly to be Hermitian. In particular, one can evaluate the components of the induced metric. Equation (3.23) reveals that $G_{12} = G_{21} = 0$ and $G_{11} = G_{22}$. The precise form of the non-trivial components is

$$G_{11} \;=\; 2g|\partial_1 F^2|^2 + \frac{1}{2}|\partial_1 G|^2 \tag{3.24}$$

with the notation $\partial_1 \equiv \frac{\partial}{\partial\sigma^1}$.

In terms of complex vielbein components, the holomorphy conditions of $W^i(\sigma)$ reduce to the following equations

$$\begin{aligned}
Re\left(e_M^1 \frac{\partial z^M}{\partial \sigma^1}\right) &= Re\left(e_M^2 \frac{\partial z^M}{\partial \sigma^2}\right) \\
Re\left(e_M^1 \frac{\partial z^M}{\partial \sigma^2}\right) &= -Re\left(e_M^2 \frac{\partial z^M}{\partial \sigma^1}\right) \\
Im\left(e_M^1 \frac{\partial z^M}{\partial \sigma^1}\right) &= -Im\left(e_M^2 \frac{\partial z^M}{\partial \sigma^2}\right) \\
Im\left(e_M^1 \frac{\partial z^M}{\partial \sigma^2}\right) &= Im\left(e_M^2 \frac{\partial z^M}{\partial \sigma^1}\right).
\end{aligned} \tag{3.25}$$

We can choose $e^1_M = 2\sqrt{g}\partial_M F^2$ and $e^2_M = \partial_M G$ which leads to the constraints

$$\begin{aligned} 2\sqrt{g}\frac{\partial F^2}{\partial \sigma^1} &= \frac{\partial \bar{G}}{\partial \sigma^2} \\ 2\sqrt{g}\frac{\partial F^2}{\partial \sigma^2} &= -\frac{\partial \bar{G}}{\partial \sigma^1} \end{aligned} \tag{3.26}$$

which are identical to our earlier results. If we now include the warp factor $2H^{-1/3}$ we had been ignoring until now and look at the full determinant of the induced metric on the M2-brane probe we conclude

$$\begin{aligned} \int \sqrt{-\det\left(G_{\mu\nu}\right)}\, dt\wedge d\sigma^1\wedge d\sigma^2 &= \int \sqrt{-2H^{-2/3}G_{00}G_{11}G_{22}}\, dt\wedge d\sigma^1\wedge d\sigma^2 \\ &= \int \left(\frac{\partial F^2}{\partial\sigma^1}\frac{\partial G}{\partial\sigma^2} - \frac{\partial G}{\partial\sigma^1}\frac{\partial F^2}{\partial\sigma^2}\right) dt\wedge d\sigma^1\wedge d\sigma^2 \\ &= \int dt\wedge dF^2\wedge dG. \end{aligned} \tag{3.27}$$

If we consider the spatial part of the probe, this induced worldvolume integral times the tension of the brane results in a probe mass given by

$$\text{Mass} = \left|\int dF^2\wedge dG\right|. \tag{3.28}$$

This gives a very natural frame in which to describe the M2-brane probe dynamics. In some sense, we have chosen the appropriate co-ordinates so the induced probe brane action has a trivial (in g) volume form. This is similar to what happened in the previous M5-brane example of the complex scalar kinetic terms (3.4).

Check from topological arguments

On a different note, one can check that the result is correct by calculating the induced Kähler form K_D and holomorphic two form Ω_D on the spatial part of the M2-brane probe. We follow closely the methods of [74] (see also [75]) which analysed the case of M2-brane and M5-brane intersections in flat space, without taking into account the full M5-brane background geometry.

Our previous results should agree with those deduced from a topological perspective. To preserve the required amount of supersymmetry, we must require the spatial volume element of the surface D to be minimised so that it saturates a topological bound. Given our choice of complex structure J' on Q^4, a useful identity is [74]

$$\frac{1}{4}\left((*K_D)^2 + |*\Omega_D|^2\right) = 1 \tag{3.29}$$

where K_D is the pullback of the Kähler form K to D, and the $*$ denotes the Hodge dual with respect to the induced metric on D.

The area A_D of the spatial part of the two-brane fulfils the inequalities

$$2A_D = 2\int_D V_D = \int_D V_D\sqrt{(*K_D)^2 + |*\Omega_D|^2} \geq \int_D |*\Omega_D|\, V_D \geq \left|\int_D \Omega_D\right| \tag{3.30}$$

where V_D is the volume-form of D. The first inequality is saturated if and only if the pullback K_D of the Kähler form vanishes, while the second condition requires that the phase of the pullback $*\Omega_D$ is constant over D. The surface D is then a holomorphic embedding with respect to some complex structure J' which is orthogonal to the complex structure J.

Explicitly, the pullback of the two-form Ω_D is

$$\begin{aligned}\Omega_D &= 2H^{-1/3}P\left(e_F \wedge e_G\right) \\ &= 2^{1/3}g^{1/6}\left(\frac{\partial F^2}{\partial\sigma^1}\frac{\partial G}{\partial\sigma^2} - \frac{\partial G}{\partial\sigma^1}\frac{\partial F^2}{\partial\sigma^2}\right) d\sigma^1 \wedge d\sigma^2.\end{aligned} \tag{3.31}$$

The area of the spatial part D of the probe is given by

$$\begin{aligned}\int\sqrt{\det\left({G_{M2}}^D\right)}\, d\sigma^1 \wedge d\sigma^2 &= \int\sqrt{G_{11}G_{22}}\, d\sigma^1 \wedge d\sigma^2 \\ &= 2^{1/3}\int g^{1/6}\left(\frac{\partial F^2}{\partial\sigma^1}\frac{\partial G}{\partial\sigma^2} - \frac{\partial G}{\partial\sigma^1}\frac{\partial F^2}{\partial\sigma^2}\right) d\sigma^1 \wedge d\sigma^2\end{aligned} \tag{3.32}$$

so we have

$$\int \Omega_D \wedge \bar{\Omega}_D = (V_D)^2 = \int \det\left({G_{M2}}^D\right). \tag{3.33}$$

We can quickly check that the induced Kähler form K_D on the spatial part of the M2-brane probe

$$
\begin{aligned}
K_D &= 2H^{-1/3}\left(2g\, dF^2 \wedge d\bar{F}^2 + 1/2\, dG \wedge d\bar{G}\right) \\
&= 0
\end{aligned} \tag{3.34}
$$

vanishes identically. This follows from the holomorphy constraints (3.26). It provides a check on the equivalence of the holomorphic two-form and the volume element of the probe as deduced from topological arguments.

Now, in order to compare Equations (3.32) and (3.28), we have to realise that the energy-momentum is an invariant quantity. The area element of the probe we have just calculated is not an invariant quantity since it does not include the time component. So we need to add a factor of $\sqrt{-g_{00}} = H^{-1/6} = 2^{-1/3} g^{-1/6}$ to Equation (3.32). Doing this then gives the invariant mass term

$$
\begin{aligned}
\text{Mass} &= 2^{1/3} \int \sqrt{-g_{00}}\, g^{1/6} \left(\frac{\partial F^2}{\partial \sigma^1} \frac{\partial G}{\partial \sigma^2} - \frac{\partial G}{\partial \sigma^1} \frac{\partial F^2}{\partial \sigma^2} \right) d\sigma^1 \wedge d\sigma^2 \\
&= \left| \int dF^2 \wedge dG \right|
\end{aligned} \tag{3.35}
$$

which then agrees with Equation (3.28).

This is also equivalent to the recent result in [55] which used a slightly different method and notation. The M2-brane probe satisfies the same calibration bound in both cases. The next section is a spinorial derivation of this bound from the supersymmetry projection conditions.

3.2.3 Spinorial derivation of the M2-brane BPS bound

General form of supersymmetry algebra for membranes

As we have mentioned in the last chapter, the understanding of the general structure of supersymmetric solutions of supergravity theories has made great strides (see for example [77] and references therein). This stems from careful analysis of the Killing spinor equations:

$$
\tilde{D}_M \epsilon = 0 \tag{3.36}
$$

where

$$\tilde{D}_M\epsilon \equiv \nabla_M\epsilon + \frac{1}{288}\left[\Gamma_M^{\ NPQR} - 8\delta_M^N\Gamma^{PQR}\right]F_{NPQR}\epsilon \tag{3.37}$$

and F is the four-form field strength of 11d supergravity.

We recall that it has proven useful to repackage $\epsilon(x)$ in terms of the following one-, two- and five-forms:

$$\begin{aligned} K_M &= \bar{\epsilon}\Gamma_M\epsilon \\ \Omega_{MN} &= \bar{\epsilon}\Gamma_{MN}\epsilon \\ \Sigma_{MNPQR} &= \bar{\epsilon}\Gamma_{MNPQR}\epsilon. \end{aligned} \tag{3.38}$$

Then $\epsilon(x)$ can be reconstructed (up to a sign) from knowledge of K, Ω and Σ. One can check that the zero-, three- and four-forms built this way and their duals vanish identically.

Following the analysis of [77], we find that one can rewrite the super-Poincaré algebra of flat eleven-dimensional supergravity coupled to a supermembrane probe

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta}P_M \pm \frac{1}{2}(C\Gamma_{MN})_{\alpha\beta}Z^{MN}, \tag{3.39}$$

where the central charge is defined to be

$$Z^{MN} = \int dX^M \wedge dX^N \tag{3.40}$$

(where the integration is taken over the spatial worldvolume of the membrane) and Q_α are the 32 component Majorana spinor charges. In shorthand notation

$$2(\epsilon Q)^2 = K_M P^M \pm \Omega_{MN}Z^{MN}. \tag{3.41}$$

Following [77], for a general curved background (without imposing any restriction on K_M), we have:

$$2(\epsilon Q)^2 = \int K_M p^M(\sigma) \pm \int (\Omega + \imath_K A). \tag{3.42}$$

In this expression, A is a three-from potential for the four-form field strength F. We have also rewritten the momentum, P^M, as an integral of the momentum density, $p^M(\sigma)$, over the spatial worldvolume of the brane. The supersymmetry algebra (3.42) leads to a BPS-type bound on the energy-momentum of the M2-brane, since $(\epsilon Q)^2 \geq 0$. We find

$$\int K_M p^M(\sigma) \geq \mp \int (\Omega + \imath_K A) \tag{3.43}$$

where the term on the RHS is topological in nature. This is indeed the topological bound we shall calculate for our M2-brane in our particular background. We shall investigate central charges in much greater depth in the next couple of chapters, for now this is all we shall require.

Calculation of the BPS bound

For clarity we restate the supergravity background metric and four-form field strength we shall use for the calculation of the topological objects constructed in the last section. These are

$$ds^2 = H^{-1/3} dx^2{}_{(1,3)} + 2H^{-1/3} g_{M\bar{N}} dz^M dz^{\bar{N}} + H^{2/3} dx^2{}_{(3)} \tag{3.44}$$

where

$$g_{M\bar{N}} \equiv 2\left(\partial_M F^2\right)\left(\overline{\partial_N F^2}\right) g + 1/2\left(\partial_M G\right)\left(\overline{\partial_N G}\right) \tag{3.45}$$

with $g = \frac{\pi N}{8\tilde{r}^3}$ and $\tilde{r} \equiv \sqrt{t^4 + |F|^4}$.

The four-form field strength is given by

$$\begin{aligned}
F_{M\bar{N}\alpha\beta} &= 2i\epsilon_{\alpha\beta\gamma}\partial_\gamma g_{M\bar{N}} \\
F_{M89(10)} &= -4i\partial_M g \\
F_{\bar{N}89(10)} &= 4i\partial_{\bar{N}} g.
\end{aligned}$$

The Hermitian metric $g_{M\bar{N}}$ can be decomposed into the tangent space zweibeins

$$\begin{aligned} e^1_M &= 2\sqrt{g}\left(\partial_M F^2\right) \\ \overline{e^2_N} &= \left(\overline{\partial_N G}\right) \end{aligned} \tag{3.46}$$

and so

$$g_{M\bar{N}} = e^a_M \left(\overline{e^b_N}\right) \delta_{a\bar{b}}. \tag{3.47}$$

Now, we saw in the previous subsection that we need to construct the quantity $\Omega = \bar{\epsilon}\hat{\Gamma}\epsilon$. We shall need the projection conditions for our particular supersymmetric configuration, which are

$$\hat{\Gamma}_{0123a\bar{b}}\epsilon = i\delta_{a\bar{b}}\epsilon \tag{3.48}$$

for the M5-brane, and

$$\mathcal{P}\epsilon = \left(e^{i\phi}\hat{\Gamma}_{0ab} + e^{-i\phi}\hat{\Gamma}_{0\bar{a}\bar{b}}\right)\epsilon = \epsilon \tag{3.49}$$

for the M2-brane, where in each case $\hat{\Gamma}$ denotes the tangent space gamma matrices. We have included an arbitrary phase for the M2-brane projection conditions, which generalises the $\phi = 0$ case of Section 3.2.2. We note that the linear combination of holomorphic and anti-holomorphic projection conditions does indeed ensure that it is an Hermitian projector with $\mathcal{P}^2 = 1$. In the next chapters we shall find the connection with structure groups and some more "hidden" structures in this background.

From the Killing spinor equation (3.37), we can also deduce that in fact

$$\epsilon(x) = H^{-1/12}\epsilon_0$$

(where ϵ_0 is a constant spinor). To find all the contributions to the two-form Ω we use the ansätze (3.38) and the aforementioned projection conditions (3.48,3.49). So for example, Ω has no contributions of the form Ω_{0a} since

$$\begin{aligned}\Omega_{0a} &= \bar{\epsilon}\hat{\Gamma}_{0a}\epsilon \\ &= -i\bar{\epsilon}\hat{\Gamma}_{123a}\epsilon \\ &= 0\end{aligned}$$

where we have used the fact that $\bar{\epsilon}\hat{\Gamma}_{123a}\epsilon$ vanishes identically and also the M5-brane projection conditions. Further, we note that it is possible to view the matrices $\hat{\Gamma}_a$ and $\hat{\Gamma}_{\bar{b}}$ as creation and annihilation operators since we have

$$\begin{aligned}\hat{\Gamma}_{a\bar{b}}\hat{\Gamma}_a\epsilon &= \delta_{a\bar{b}}\hat{\Gamma}_a\epsilon \\ \hat{\Gamma}_{a\bar{b}}\hat{\Gamma}_{\bar{b}}\epsilon &= -\delta_{a\bar{b}}\hat{\Gamma}_{\bar{b}}\epsilon.\end{aligned} \tag{3.50}$$

Using these relations, we find that the only non-vanishing components of the two-form Ω give

$$\Omega = -H^{-1/6}\left(e^{-i\phi}\epsilon_{\mu\nu}dz^\mu \wedge dz^\nu + e^{i\phi}\epsilon_{\bar{\mu}\bar{\nu}}d\bar{z}^\mu \wedge d\bar{z}^\nu\right) + \dots . \tag{3.51}$$

The rest of the terms do not involve the hyper-Kähler manifold Q^4, but we shall have more to say about them in Chapter 5. Our normalisation was chosen such that $\epsilon^\dagger\epsilon = H^{-1/6}$. We should also note that the tensors $\epsilon_{\mu\nu}$ include a factor of $H^{-1/3}$ coming in from the warp factor of the metric (3.17). Now rewriting the above in terms of w, y and F^2, G we find

$$\begin{aligned}\Omega &= -H^{-1/2}\left(e^{-i\phi}e^1_{[1}e^2_{2]}dz^1 \wedge dz^2 + e^{i\phi}\bar{e}^1_{[1}\bar{e}^2_{2]}d\bar{z}^1 \wedge d\bar{z}^2\right) \\ &= 1/2\left(e^{-i\phi}dw \wedge dy + e^{i\phi}d\bar{w} \wedge d\bar{y}\right) = 1/2\left(e^{-i\phi}dF^2 \wedge dG + e^{i\phi}d\bar{F}^2 \wedge d\bar{G}\right)\end{aligned} \tag{3.52}$$

using the fact $H = 4g$ in our conventions and also the condition $(\partial_y F^2)(\partial_w G) - (\partial_w F^2)(\partial_y G) = 1$.

Finally, we also note that the inner product $\imath_K A$ does not give any contributions for our choice of background. This is because the Killing vector K only has non-trivial components in the $(0,1,2,3)$ space, whereas the three-form potential A only has components in the $(4,5,6,7,8,9,10)$ space, so the inner product vanishes.

Therefore, the BPS lower bound on the mass of the M2-brane in our particular background is given by

$$\int K_M P^M \geq \mp \int \Omega \tag{3.53}$$

in accordance with Equation (3.43). This also reproduces our earlier result of Equation (3.28) if we note that under an appropriate worldvolume co-ordinate definition, setting our phase $\phi = 0$ and using Equation (3.26) we would have

$$1/2\left(dF^2 \wedge dG + d\bar{F}^2 \wedge d\bar{G}\right) = dF^2 \wedge dG.$$

We note that there are no supergravity corrections to the holomorphic two-form which gives the mass of the BPS monopoles in the dual gauge theory. All the Seiberg-Witten analysis then follows through unchanged.

3.3 The $\mathcal{N}=2$ Super Yang-Mills theory

We will now show how the supergravity solution (2.27), along with the known form of the M5-brane worldvolume action (2.44), can be used to extract information about the corresponding gauge theory. We will study the dynamics of an M5-brane probe which wraps around the M-theory direction, and thus reduces to a D4-brane upon dimensional reduction. We will also calculate the Yang-Mills coupling and the theta angle for the $\mathcal{N}=2$ gauge theory living on the D4-brane worldvolume. For a similar analysis in the Type IIB picture see [62].

3.3.1 Reduction process for the M5-brane worldvolume action

The first step is to dimensionally reduce the M5-brane worldvolume action along the M-theory direction to arrive at the D4-brane action. This is actually a two-step process, as a direct dimensional reduction yields the so-called dual D4-brane action. So after performing the reduction, we then have to dualise the resulting action to arrive at the usual string frame DBI action for the D4-brane. We will use and follow the analysis of [79] for these steps, and refer the reader there for further details.

It is important to note that we will be using a modified Kaluza-Klein reduction ansätze. Explicitly, the eleven-dimensional metric can be expressed in component form as

$$G_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{-2\phi/3}\left(g_{\mu\nu} + C_\mu C_\nu\right) & v e^{\phi/3} C_\mu \\ v e^{\phi/3} C_\nu & v^2 e^{4\phi/3} \end{pmatrix} \tag{3.54}$$

where v is the winding number, giving the number of times the M5-brane wraps the compact dimension and C_μ is the R-R one-form. For the M5-brane worldvolume reduction we shall set $v = 1$. We can rewrite the M5-brane action in the form

$$\begin{aligned} S_{M5} = -\tau_5 \int d^6\sigma \Big[& \sqrt{-G_6}\sqrt{1 + \hat{z}_1 + \hat{z}_1^2/2 - \hat{z}_2} \\ & + \frac{1}{24(\partial a)^2}\epsilon^{\hat{l}\hat{m}\hat{n}\hat{p}\hat{q}\hat{r}}\hat{\mathcal{H}}_{\hat{p}\hat{q}\hat{r}}\hat{\mathcal{H}}_{\hat{m}\hat{n}\hat{s}}g^{\hat{s}\hat{t}}\partial_{\hat{l}}a\partial_{\hat{t}}a + \mathcal{L}_{WZ}\Big] \end{aligned} \tag{3.55}$$

where we have denoted the worldvolume co-ordinates by $\sigma^{\hat{\mu}} = (\sigma^\mu, \sigma^5)$ with $\mu = 0,1,2,3,4$ and G_6 is the six-dimensional determinant. The z variables are defined to be

$$\hat{z}_1 = \frac{\mathrm{tr}(\hat{\tilde{\mathcal{H}}}^2)}{2} = \frac{G_{\hat{\mu}\hat{\nu}}\hat{\tilde{\mathcal{H}}}^{\hat{\nu}\hat{\rho}}G_{\hat{\rho}\hat{\lambda}}\hat{\tilde{\mathcal{H}}}^{\hat{\lambda}\hat{\mu}}}{2} \tag{3.56}$$

$$\hat{z}_2 = \frac{\mathrm{tr}(\hat{\tilde{\mathcal{H}}}^4)}{4} = \frac{\hat{G}\hat{\tilde{\mathcal{H}}}\hat{G}\hat{\tilde{\mathcal{H}}}\hat{G}\hat{\tilde{\mathcal{H}}}\hat{G}\hat{\tilde{\mathcal{H}}}}{4}. \tag{3.57}$$

If we now fix the gauge so that the compactified direction is taken to be the $a = \sigma^5$ direction (and hence $\partial_{\hat{\mu}}a = \delta^5_{\hat{\mu}}$), then we find that the quantity

$$(\partial a)^2 = G^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}a\partial_{\hat{\nu}}a \tag{3.58}$$

reduces to G^{55}, and both G^{55} and $G^{\rho 5}$ are components of the six-dimensional inverse metric $G^{\mu\nu}$. As direct calculation shows, these are given by $G^{55} = (1 + C^2)$ and $G^{\rho 5} = -e^{\phi}C^{\rho}$.

Upon dimensional reduction, the second term in the M5-brane action (3.55) above splits into two, in particular

$$\mathcal{L}_{\mathcal{H}\tilde{\mathcal{H}}} \longrightarrow \tau_5 \left(-\frac{\epsilon_{\mu\nu\lambda\sigma\tau}}{8(1+C^2)} e^{\phi} C^{\mu} \tilde{\mathcal{H}}^{\nu\lambda} \tilde{\mathcal{H}}^{\sigma\tau} + \frac{1}{24} \epsilon^{\mu\nu\rho\lambda\sigma(5)} \mathcal{H}_{\rho\lambda\sigma} \mathcal{H}_{\mu\nu(5)} \right) \tag{3.59}$$

where now the second term above contributes to the original Wess-Zumino term to form a new term WZ' (see [66] for more details). We have used the explicit expressions for G^{55} and $G^{\mu 5}$ in the first term.

Fixing the gauge and dimensionally reducing the DBI term yields

$$\mathcal{L}_{DBI} \longrightarrow \tau_5 e^{-\phi} \sqrt{-G_5} \sqrt{1 + e^{2\phi} z_1 + e^{4\phi}(z_1^2/2 - z_2)} \tag{3.60}$$

where

$$\begin{aligned} z_1 &= \tfrac{1}{2}\mathrm{tr}\left(\tilde{\mathcal{H}}^2\right) \\ z_2 &= \tfrac{1}{4}\mathrm{tr}\left(\tilde{\mathcal{H}}^4\right) \end{aligned}$$

and the dimensional reduction of the field $\hat{\tilde{\mathcal{H}}}$ is given by the expression:

$$\hat{\tilde{\mathcal{H}}} \rightarrow e^{\frac{10}{3}\phi} \tilde{\mathcal{H}}. \tag{3.61}$$

We have used the fact that $\hat{z}_1 \rightarrow e^{2\phi} z_1$ and $\hat{z}_2 \rightarrow e^{4\phi} z_2$

We will now rescale the Φ and $\mathcal{H}$ fields to absorb the factor of the M5-brane tension τ_5 in front of the DBI and $\mathcal{H}\tilde{\mathcal{H}}$ term. This will then put our action in the same form as that of [79], and their dualisation procedure follows trivially. The rescalings are of the form

$$e^{\phi'} = e^{\phi - \ln \tau_5} \tag{3.62}$$

$$\tilde{\mathcal{H}}' = \tau_5 \tilde{\mathcal{H}}, \tag{3.63}$$

so we see that the combinations $e^{2\phi} z_1$ and $e^{4\phi} z_2$ appearing inside the square root above are actually invariant under this rescaling.

So grouping together the various terms we can rewrite our compactified M5-brane worldvolume action, which is actually the dual D4-brane action, as

$$S^*_{D4} = -\int d^5\sigma \left(e^{-\phi'}\sqrt{-G_5}\sqrt{1+e^{2\phi'}z_1' + e^{4\phi'}(z_1'^2/2 - z_2')} \right. \quad (3.64)$$

$$\left. + \frac{\epsilon_{\mu\nu\lambda\sigma\tau}}{8(1+C^2)} e^{\phi'} C^{\mu} \tilde{\mathcal{H}}'^{\nu\lambda} \tilde{\mathcal{H}}'^{\sigma\tau} \right) + \int \mathcal{L}_{WZ'} \quad (3.65)$$

with the modified Wess-Zumino term given by

$$WZ' = e^{-\phi'} C^{(5)} + \frac{1}{2}\tilde{\mathcal{H}}' \wedge C^{(3)} + \frac{1}{4}\tilde{\mathcal{H}}'^{\mu\nu}\partial_{(5)} B_{\mu\nu}. \quad (3.66)$$

It now follows from the analysis of [79] that the D4-brane action with a constant dilaton background field is given by

$$S_{D4} = -\int d^5\sigma\, e^{-\phi'}\sqrt{-\det(G_{\mu\nu} + \mathcal{F}_{\mu\nu})} \quad (3.67)$$

$$-\int e^{-\phi'} \left(C_{(5)} + C_{(3)} \wedge \mathcal{F} + \frac{1}{2} C_{(1)} \wedge \mathcal{F} \wedge \mathcal{F} \right). \quad (3.68)$$

The two are related by

$$-\frac{\delta S_{D4}}{\delta F_{\mu\nu}} = \tilde{H}^{\mu\nu} \quad (3.69)$$

where we note that we have the six-dimensional Hodge duals $\tilde{H} = *H$ and the definitions $\hat{\mathcal{H}} \equiv H - e^{-\phi} C^{(3)}$ and $\mathcal{F} = F - b_{(2)}$. The method of [79] relies on constructing Lorentz invariant quantities with a particularly simple choice for the form of $F_{\mu\nu}$, which is then used to solve Eq. (3.69). Since the quantities are Lorentz invariant, it is straightforward to pass from this special frame to a more general frame.

3.3.2 Dimensional reduction of the background supergravity solution

Before we proceed any further, we also need to dimensionally reduce the background supergravity solution (3.44) down to the ten-dimensional Type IIA string frame metric. We recall that we are using a modified Kaluza-Klein reduction ansätze which, expressed as a line element, has the form

$$
\begin{aligned}
ds^2_{(1,10)} &= e^{-2\phi/3} ds^2_{(1,9)} + e^{4\phi/3}\left(dx^7 + e^{-\phi} C_\mu dx^\mu\right)^2 && (3.70)\\
\hat{F}_{(4)} &= \mathcal{F}_{(4)} + \mathcal{T}_{(3)} \wedge dx^7 && (3.71)
\end{aligned}
$$

where $\hat{F}_{(4)} = dC_{(3)}$ is the field strength for the background three-form potential $C_{(3)}$, with $\mathcal{F}_{(4)}$ and $\mathcal{T}_{(3)} = dI_{(2)}$ being the RR four-form and the NSNS three-form field strengths of the ten-dimensional Type IIA theory. We recall that the coordinate x^7 is the circle (of radius R) we are compactifying on, with periodicity $2\pi R$.

For clarity and ease of reading, we write down the eleven-dimensional supergravity solution we stated earlier.

$$
ds^2 = H^{-1/3} dx^2{}_{(1,3)} + 2H^{-1/3} g_{M\bar{N}} dz^M dz^{\bar{N}} + H^{2/3} dx^2{}_{(3)} \tag{3.72}
$$

with $M, N = F^2, G$ and where we recall that $F^2(w,y)$ and $G(w,y)$ are holomorphic functions of w, y with

$$
\begin{aligned}
w &= \tfrac{v}{l_s{}^2} = \tfrac{vR}{l_P{}^3}\\
t^2 &= \tfrac{r}{g_s l_s{}^3} = \tfrac{r}{l_P{}^3}\\
y &= \tfrac{s}{R}
\end{aligned}
$$

and

$$
\begin{aligned}
v &= x^4 + ix^5\\
s &= x^6 + ix^7.
\end{aligned} \tag{3.73}
$$

We can rewrite the complex Hermitian metric $g_{M\bar{N}}$ in terms of real co-ordinates, and then use the Hermiticity condition to simplify it further. Our aim is to calculate field theory quantities on the resulting D4-brane worldvolume action, in which case the endpoints of the D4-brane are allowed to have different fixed values in the x^7 direction. With this in mind, we modify our original co-ordinates

$$
\begin{aligned}
x^6 &= \tilde{x}^6 \cos\theta\\
x^7 &= \tilde{x}^7 + \tilde{x}^6 \sin\theta
\end{aligned} \tag{3.74}
$$

so that we are effectively tilting in the x^7 direction (see Fig. 3.1). One may consider that this angle is not an arbitrary paramter of the probe but rather that it is fixed by the background, see Eq. (2.54). A different value of θ would therefore correspond to a different background and corresponding gauge theory. We consider only constant θ values. In terms of these new co-ordinates, we may write

$$dy = (e^{i\theta} d\tilde{x}^6 + i d\tilde{x}^7)/R. \tag{3.75}$$

The resulting eleven-dimensional metric is then given by

$$ds_{11}^2 = H^{-1/3} dx_{(1,3)}^2 + 2H^{-1/3} M_{\mu\nu} dx^\mu dx^\nu + 2H^{-1/3} ds_{KK}^2 + H^{2/3} dx_\perp^2 \tag{3.76}$$

with the Kaluza-Klein part of the metric being

$$ds_{KK}^2 = g_{y\bar{y}} \left(d\tilde{x}^7 + D_\mu dx^\mu \right)^2. \tag{3.77}$$

The notation for $g_{y\bar{y}}$ refers to

$$g_{y\bar{y}} = H/2 \left(\partial_y F^2\right) \left(\overline{\partial_y F^2}\right) + 1/2 \left(\partial_y G\right) \left(\overline{\partial_y G}\right) \tag{3.78}$$

If we denote the $\mu\nu = (v, \tilde{6})$, then the $M_{\mu\nu}$ part corresponds to

$$\begin{aligned} M_{\mu\nu} dx^\mu dx^\nu &= g_{y\bar{y}} \cos^2\theta d\tilde{x}^6 + \cos\theta (g_{v\bar{y}} dv + g_{y\bar{v}} d\bar{v}) d\tilde{x}^6 \\ &\quad + g_{v\bar{v}} dv d\bar{v} - \frac{1}{4 g_{y\bar{y}}} (g_{v\bar{y}} dv - g_{y\bar{v}} d\bar{v})^2 \end{aligned} \tag{3.79}$$

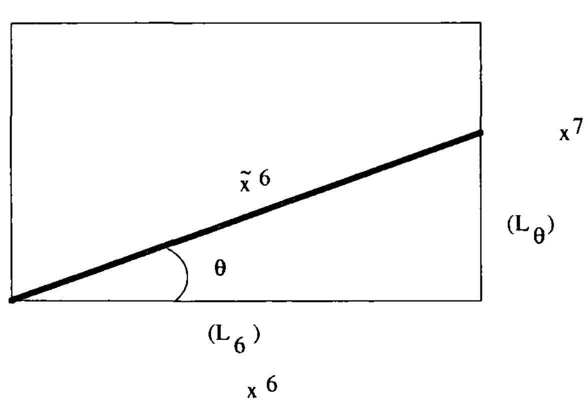

Figure 3.1: The tilted D4-brane with endpoints at different values of x^7.

and D_μ is in general given by

$$D_\mu dx^\mu = -\frac{i}{2g_{y\bar{y}}}(g_{v\bar{y}}dv + g_{y\bar{v}}d\bar{v}) + \sin\theta d\tilde{x}^6. \tag{3.80}$$

3.3.3 The Yang-Mills coupling and theta angle

Having arrived at the expression for the D4-brane action (3.68), we now show how the supergravity solution (3.44) can be used to extract information about the corresponding gauge theory.

We wish to study the dynamics of the $SU(N)$ gauge fields that propagate on the wrapped M5-branes. We will look at Hanany-Witten type configurations where the D4-brane is finite in extent in the $\tilde{x}^6$ direction, which effectively reduces the worldvolume degrees of freedom to four. This four-dimensional part is flat, and is the Minkowski space-time $\mathbf{R}^{(1,3)}$ where a supersymmetric gauge theory with eight supercharges is defined. One finds that the low-energy four-dimensional gauge theory is a pure $\mathcal{N}=2$ SYM theory with gauge group $SU(N)$.

The action for our M5-brane probe reduced to a D4-brane was calculated in the last section to be

$$\begin{aligned} S_{D4} &= -\int d^5\sigma\, e^{-\phi'}\sqrt{-\det(G_{mn}+\mathcal{F}_{mn})} && (3.81)\\ &\quad -\int e^{-\phi'}\left(C_{(5)} + C_{(3)}\wedge\mathcal{F} + \frac{1}{2}C_{(1)}\wedge\mathcal{F}\wedge\mathcal{F}\right) && (3.82)\end{aligned}$$

where all the bulk fields are understood to be the pullbacks onto the brane worldvolume which is parametrised by $\sigma = (x^0, x^1, x^2, x^3, \tilde{x}^6)$. By expanding the square root part of the above action and examining the component which is quadratic in the field strength $\mathcal{F}$ we can deduce the Yang-Mills coupling for the worldvolume theory. We achieve this by promoting $\mathcal{F}$ to an $SU(N)$ field and by giving it an adjoint index A. If the generators are normalised in such a way that $\mathrm{tr}\left(T^A T^B\right) = (1/2)\delta^{AB}$ for the fundamental representation, then the above procedure leads to

$$S_{YM} = -\frac{1}{g_{YM}^2}\int d^4\sigma \frac{1}{4}\mathcal{F}^A_{\alpha\beta}\mathcal{F}^{\alpha\beta}_A + \frac{(\Theta_{YM}+2\pi n)}{32\pi^2}\int d^4\sigma \mathcal{F}^A_{\alpha\beta}\tilde{\mathcal{F}}^{\alpha\beta}_A, \tag{3.83}$$

where

$$\frac{1}{g_{YM}^2} = \frac{\tau_5}{2}\left(2\pi\alpha'\right)^2 \int d\tilde{x}^6\, d\tilde{x}^7\, e^{-\phi}\sqrt{-G_5}\left(g^{mn}\right)^2 \tag{3.84}$$

for $(m,n) = 0,1,2,3$ (i.e. $(g^{mn})^2$ refers to two factors of the inverse metric in the $0,1,2,3$ flat space) and $\sqrt{-G_5}$ denotes the square-root determinant of the induced worldvolume metric.

We can also deduce the value of the Yang-Mills theta angle to be

$$\Theta_{YM} = \left[\tau_5(2\pi)^2\left(2\pi\alpha'\right)^2 \int d\tilde{x}^6\, d\tilde{x}^7\, e^{-\phi}C_6^{(1)}\right] \mod 2\pi \tag{3.85}$$

where $C_6^{(1)}$ denotes the six component of the one form $C^{(1)}$.

Now, to calculate the pullback of the various forms we need to establish the ten-dimensional background metric in the string frame. We can do this by comparing our eleven-dimensional metric (3.76) with our Kaluza-Klein reduction ansätze metric

$$ds^2_{(1,10)} = e^{-2\phi/3}ds^2_{(1,9)} + e^{4\phi/3}\left(d\tilde{x}^7 + e^{-\phi}C_\mu dx^\mu\right)^2 \tag{3.86}$$

for $\mu = v, \tilde{x}^6$. Due to the Hermitian nature of the metric, the D_6 component of the eleven-dimensional metric simplifies to

$$D_6 = +\sin\theta \tag{3.87}$$

where $D_6 = H^{1/4}\left[8g_{y\bar{y}}\right]^{-1/4}C_6$.

We can then read off the R-R (C_μ) and NS-NS (ϕ, $g_{\mu\nu}$) fields from the dimensional reduction of the background metric (3.76). We find the dilaton is given by

$$\phi = \frac{3}{4}\ln\left(\frac{2g_{y\bar{y}}}{H^{1/3}}\right), \tag{3.88}$$

and for the six component of the R-R one form we get

$$C_6^{(1)} = H^{-1/4}\sin\theta\left[8g^3_{y\bar{y}}\right]^{(1/4)}. \tag{3.89}$$

The ten-dimensional string frame background metric is given by

$$ds_{10}^2 = \sqrt{2g_{y\bar{y}}}\left[H^{-1/2}dx_{(1,3)}^2 + 2H^{-1/2}M_{\mu\nu}dx^\mu dx^\nu + H^{1/2}dx_\perp^2\right]. \tag{3.90}$$

If we now place our probe so that it lies along the 0123[$\tilde{6}$] directions, we can calculate the induced metric and thus the Yang-Mills coupling. If we take into account the possibility of our probe wrapping the x^7 direction N times, this corresponds to looking at an SU(N) gauge theory instead of a U(1) gauge theory. The coupling turns out to be

$$\frac{1}{g_{YM}^2} = \frac{N}{8\pi^2 g_s l_s}\int d\tilde{x}^6\ \cos\theta \tag{3.91}$$

where to evaluate the determinant for the induced worldvolume metric we have used the result $M_{\tilde{6}\tilde{6}} = g_{y\bar{y}}\cos^2\theta$, which follow from the Hermitian condition of the metric components. The theta angle turns out to be quite simple as well, explicitly

$$\Theta_{YM} = \left[\frac{N}{g_s l_s}\int d\tilde{x}^6\ (\sin\theta)\right] \quad \mathrm{mod}\ 2\pi N. \tag{3.92}$$

So the end result is that the usual Seiberg-Witten analysis goes through unchanged. The bending of the NS5-branes given by the Seiberg-Witten curve is encoded in the $\tilde{x}^6$ integral. In particular, Witten showed how this bending actually corresponds to the logarithmic running of the Yang-Mills gauge coupling. In particular, the separation of the NS5-branes is logarithmic in $|v|$ since the end of the D4-brane in the NS5-brane is of co-dimension 2. Essentially, the integral

$$L_6(v) = \int d\tilde{x}^6 \cos\theta \sim \ln|v|$$

is governed by the limits of integration at the D4-brane endpoints. The theta angle can be seen to be fixed since θ is constant and also $\int d\tilde{x}^6 \sin\theta = \int(dx^7 - d\tilde{x}^7)$ which is constant since the difference in x^7 of the endpoints is kept fixed.

As we saw in Chapter 2, the background NS5-branes can be separated in the x^7 direction by an arbitrary phase, so we can think of our D4-brane probe as being tilted by an angle θ which corresponds to the angle that the NS5-branes are separated by. Different values of this angle would therefore correspond to the theta angle of different gauge theories as they reflect a different backgrounds.

Since supersymmetric D4-brane probes of the type we are considering are restricted to lie in $\mathbf{C}^2$ to end on an NS5-brane, this also restricts the holographic radial co-ordinate to $|v|$, since that is the scope of movement of the probe in a radial direction. In eleven dimensions, this probe picture is an approximation since an M5-brane probe cannot be both holomorphic and end on the background branes due to the smoothing out which occurs.

For the canonical example of the Hanany-Witten Type IIA model with two NS5-branes separated by a distance L_6 and the D4-brane endpoints on the NS5-branes separated in the x^7 direction by a distance L_θ, the gauge coupling and theta angle reduce to the classical values

$$\begin{aligned}\frac{1}{g_{YM}^2} &= \frac{L_6}{8\pi^2 g_s l_s} \\ \Theta_{YM} &= \frac{L_\theta}{g_s l_s}\end{aligned}$$

and the $\mathcal{N}=2$ complex gauge coupling can be written in the usual form

$$\tau = \frac{\Theta_{YM}}{2\pi} + i\frac{4\pi}{g_{YM}^2}. \tag{3.93}$$

3.3.4 Instantons

We can also show that instantons are correctly represented as Euclidean D0-branes living on the colour D4-branes [80,81] in the $\tilde{x}^6$ direction (for a similar analysis in Type IIB see [62]). The worldvolume action of a Euclidean D0-brane in our special frame is given by

$$S_{D0} = \tau_0 \int d\tilde{x}^6\, e^{-\phi}\sqrt{G_{\tilde{6}\tilde{6}}} - i\tau_0 \int e^{-\phi} C_6^{(1)} \tag{3.94}$$

where

$$\tau_0 = \frac{1}{g_s l_s}, \tag{3.95}$$

and the bulk fields are understood to be the pullbacks onto the brane worldvolume. The appearance of i is due to the Wick rotation we perform to arrive at the Euclidean action.

Using the ten-dimensional metric we computed earlier (3.86), as well as the dilaton (3.88), the R-R one-form (3.89) and the explicit form of the metric (3.79) it is easy to see that

$$S_{D0} = \frac{8\pi^2}{g_{YM}^2} - i\Theta_{YM} \tag{3.96}$$

which is the correct form of the instanton action. We have used the previous expressions for the Yang-Mills coupling (3.91) and theta angle (3.92) to arrive at this result. So we conclude that the gauge theory instantons of the $\mathcal{N}=2$ SYM theory are indeed represented by Euclidean D0-branes extended in the $\tilde{x}^6$ direction, as one should expect from general considerations.

3.4 The $\mathcal{N}=1$ Super Yang-Mills theory

In this section we shall use an M5-brane probe in a background of an M5-brane wrapped on a Riemann surface Σ in three complex dimensions. As mentioned previously, this is the eleven-dimensional supergravity dual of certain $\mathcal{N}=1$ field theories (so-called MQCD theories [14,56]). We study the dynamics of an M5-brane probe that wraps around the M-theory direction, and thus becomes a D4-brane upon dimensional reduction. We will also calculate the Yang-Mills coupling and the theta angle for the $\mathcal{N}=1$ gauge theory living on the D4-brane worldvolume. For a similar analysis from the Type IIB viewpoint see also [62,63].

To begin with, however, we shall look at an alternative method of deriving the supergravity solution for this wrapped M5-brane background. We shall employ the bilinear spinor formalism introduced in Chapter 2, along with their corresponding differential equations, to determine the components of the field strength and the exact form of the metric. This approach is similar to the one employed in [30], where this supergravity solution was also derived.

3.4.1 The supergravity solution

The supersymmetry preserving solutions of eleven-dimensional supergravity relevant for describing the M5-brane setup were described in [31]. The method is very similar to the $\mathcal{N}=2$ case, so we shall go straight to the results. We recall the form of the solution:

$$\begin{aligned} ds^2 &= H^{-1/3}dx^2{}_{(1,3)} + 2H^{1/6}g_{M\bar{N}}dz^M dz^{\bar{N}} + H^{2/3}dy^2 && (3.97)\\ \det g &= H \\ F &= \partial_y(\omega\wedge\omega) - i\partial(H^{1/2}\omega)\wedge dy + i\bar{\partial}(H^{1/2}\omega)\wedge dy && (3.98)\\ \bar{\partial}(\omega\wedge\omega) &= 0 = \partial(\omega\wedge\omega). && (3.99)\end{aligned}$$

In the above equations, the z^M are holomorphic coordinates:

$$\begin{aligned} z^1 &= v = x^4 + ix^5 \\ z^2 &= w = x^6 + ix^7 \\ z^3 &= s = x^8 + ix^9. \end{aligned}$$

The metric (2.38) is of the form $\mathbf{R}^{(1,3)} \times M_6 \times \mathbf{R}^{(1)}$, where M_6 is a three complex-dimensional Hermitian manifold and $y = x^{(10)}$ denotes the remaining totally transverse direction. Also, ∂ denotes the $(1,0)$ exterior derivative $\partial = dz^M\partial_M$ in $\mathbf{C}^3$. The metric tensor $g_{M\bar{N}}$ is Hermitian, a property we shall use in the following calculations. It has an associated Hermitian 2-form

$$\omega = ig_{M\bar{N}}dz^M \wedge dz^{\bar{N}} \tag{3.100}$$

which is useful in expressing the field strength F in a more elegant form. One can check that the $\mathcal{N}=2$ solution satisfies the above constraints.

An alternative method

An alternative and fairly straightforward method of finding the supergravity solution is applying ideas from the recent work on the classification of supersymmetric solutions of eleven-dimensional supergravity [24] (see also [30]). We shall demonstrate a derivation of the supergravity solution using these ideas.

Firstly, we recall once more that the spinor $\epsilon(x)$ which satisfies the Killing spinor equation (2.2) can be reconstructed (up to a sign) from the following one-, two- and five-forms:

$$\begin{aligned} K_M &= \bar{\epsilon}\Gamma_M\epsilon \\ \Omega_{MN} &= \bar{\epsilon}\Gamma_{MN}\epsilon \\ \Sigma_{MNPQR} &= \bar{\epsilon}\Gamma_{MNPQR}\epsilon. \end{aligned} \tag{3.101}$$

One can check that the zero-, three- and four-forms built in a similar way vanish identically.

Furthermore, as recent work on G-structures and related ideas has brought to the fore, if we start with a D=11 geometry with a $Spin(10,1)$ structure and assume that we have a globally defined spinor, then, at a point, the isotropy group of the spinor is known to be either $SU(5)$ or $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ depending on whether K is time-like or null, respectively.

Using the fact that our particular background preserves four Killing spinors, we can always consider the case where K is null. The forms K, Ω and Σ then define a $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure corresponding to the stability group of the spinor ϵ. A possible set of tangent space projection conditions for the spinor $\epsilon(x)$ is given in this case by:

$$\hat{\Gamma}_{0123 z_a \bar{z}_b}\epsilon = i\delta_{a\bar{b}}\epsilon. \tag{3.102}$$

However, for explicit calculations, we can choose an arbitrary spinor satisfying this constraint by choosing further appropriate projection conditions. A compatible projection choice along the 01 directions would be

$$\hat{\Gamma}_{01}\epsilon = \pm\epsilon \tag{3.103}$$

where the ambiguity of sign comes from the requirement that the projector squares to 1. It should be emphasised that the equations for 1/8-SUSY hold for arbitrary $\pm$ sign of this projection, all that is required is that Equation (3.102) is satisfied.

Similarly, for the z_1, z_2, z_3 space, we may use this freedom to choose the compatible projection condition

$$1/8\left(e^{i\phi}\hat{\Gamma}_{2z_1z_2z_3}+e^{-i\phi}\hat{\Gamma}_{2\overline{z_1}\overline{z_2}\overline{z_3}}\right)\epsilon=\epsilon \tag{3.104}$$

and we can check this projector is also Hermitian, as is required. Again, the equations for 1/8-SUSY hold for arbitrary phase ϕ.

Note that using the identity $\hat{\Gamma}_{0123456789y}\equiv 1$ we can show that our projections imply:

$$\hat{\Gamma}_y\epsilon=-\epsilon. \tag{3.105}$$

These provide a set of independent, commuting projections which determine a unique spinor up to scale. The scale of the spinor is given by fixing

$$\epsilon^{\dagger}\epsilon=\Delta. \tag{3.106}$$

We will determine the value of Δ shortly.

To calculate the forms and solve the differential equations for the field strength, we first need to determine the form of the metric. We shall start with a metric ansätze of the form $\mathbf{R}^{(1,3)}\times\mathbf{C}^3\times\mathbf{R}^{(1)}$ with the assumption that the complex space is Hermitian. This is typical of M5-branes wrapping 2-cycles in $\mathbf{C}^3$. In general we can have

$$ds^2=f^2dx_{(1,3)}+2e^a_M\left(\overline{e^b_N}\right)\delta_{a\bar{b}}dz^Mdz^{\overline{N}}+g^2dy^2 \tag{3.107}$$

with f,g arbitrary functions of $z^M, z^{\overline{N}}$ and y. As before we let a,b run through $1,2,3$ and normalise the complex part of the metric such that $\delta_{1\bar{1}}=1/2$.

We can now proceed to calculate the non-trivial components of each form. A quick calculation reveals that the $K_i (i=2,3), K_a, K_{\bar{b}}(a,b=1,2,3)$ and K_y components vanish since, for example,

$$\hat{K}_y \;=\; \bar{\epsilon}\hat{\Gamma}_y\epsilon \;=\; -\bar{\epsilon}\left(\hat{\Gamma}_y\right)^2\epsilon \;=\; -\bar{\epsilon}\epsilon \;=\; 0$$

where in the second step we have used the $\hat{\Gamma}_y$ projection condition (3.105), and in the last step the fact that $\bar{\epsilon}\epsilon$ vanishes identically.

The value of Δ can be determined taking advantage of the fact that K is a Killing vector. A brief calculation shows that this Killing vector is given by

$$K = -\Delta f \left(dt \mp dx^1\right). \tag{3.108}$$

Its defining property is that the Lie derivative of our metric ansätze with respect to this vector should vanish. This yields a number of constraints which collectively imply

$$\partial_q \left(\frac{f}{\Delta}\right) = 0 \Longrightarrow \frac{f}{\Delta} = constant \tag{3.109}$$

with q running over all the spacetime co-ordinates $(0 \ldots y)$. In our normalisation we set this constant equal to one which fixes the value of Δ to be

$$\Delta = f = \sqrt{-g_{00}}. \tag{3.110}$$

We can proceed in a similar fashion to determine the non-trivial components of the two- and five-forms. From our metric ansätze we can compute the relevant components. We find

$$\begin{aligned}
K &= -f^2\left(dt \mp dx^1\right) && (3.111)\\
\Omega &= f^2 g\left(dt \mp dx^1\right)\wedge dy && (3.112)\\
\Sigma &= \mp i f^4 e^a_{[M}e^{\bar{b}}_{\bar{N}]}\delta_{a\bar{b}}\left(dt \mp dx^1\right)\wedge dx^2\wedge dx^3\wedge dz^M\wedge dz^{\bar{N}} \\
&\quad - f^2 e^a_{[M}e^{\bar{b}}_{\bar{N}}e^c_P e^{\bar{d}}_{\bar{Q}]}\delta_{a\bar{b}}\delta_{c\bar{d}}\left(dt \mp dx^1\right)\wedge dz^M\wedge dz^{\bar{N}}\wedge dz^P\wedge dz^{\bar{Q}} \\
&\quad + \frac{e^{-i\phi}}{16} f^3 e^1_{[1}e^2_2 e^3_{3]}\left(dt \mp dx^1\right)\wedge dx^2\wedge dz^1\wedge dz^2\wedge dz^3 \\
&\quad + \frac{e^{i\phi}}{16} f^3 e^{\bar{1}}_{[\bar{1}}e^{\bar{2}}_{\bar{2}} e^{\bar{3}}_{\bar{3}]}\left(dt \mp dx^1\right)\wedge dx^2\wedge dz^{\bar{1}}\wedge dz^{\bar{2}}\wedge dz^{\bar{3}} \\
&\quad \pm i\frac{e^{-i\phi}}{16} f^3 e^1_{[1}e^2_2 e^3_{3]}\left(dt \mp dx^1\right)\wedge dx^3\wedge dz^1\wedge dz^2\wedge dz^3 \\
&\quad \mp i\frac{e^{i\phi}}{16} f^3 e^{\bar{1}}_{[\bar{1}}e^{\bar{2}}_{\bar{2}} e^{\bar{3}}_{\bar{3}]}\left(dt \mp dx^1\right)\wedge dx^3\wedge dz^{\bar{1}}\wedge dz^{\bar{2}}\wedge dz^{\bar{3}}. && (3.113)
\end{aligned}$$

Now $\epsilon(x)$ being a Killing spinor also implies that K, Ω and Σ satisfy a set of differential equations. These were given in [24]:

$$
\begin{aligned}
dK &= \frac{2}{3}\iota_\Omega F + \frac{1}{3}\iota_\Sigma * F && (3.114)\\
d\Omega &= \iota_K F && (3.115)\\
d\Sigma &= \iota_K * F - \Omega \wedge F. && (3.116)
\end{aligned}
$$

We now need to solve for the field strength F. In the process we shall see that the form of the metric is also fully determined by this set of equations.

If we begin by studying the consequences from the differential equation for Ω (3.115), we quickly find that, for example,

$$
\begin{aligned}
\iota_K F_{01\alpha\beta} &= \mp G(x^i, z^M, z^{\bar{N}}, y)\left(dt \mp dx^1\right) \wedge d\alpha \wedge d\beta \\
d\Omega &= \partial_\chi \left(f^2 g\right)\left(dt \mp dx^1\right) \wedge dy \wedge d\chi.
\end{aligned}
$$

Setting $\alpha = y$ and $\beta = \chi$ we have

$$
\mp G = \partial_\chi \left(f^2 g\right) \quad \text{for } \chi = z^M, z^{\bar{N}}.
$$

This implies $G = 0, \partial_\chi \left(f^2 g\right) = 0 \Longrightarrow f^2 g h(y) = constant$, where $h(y)$ is an arbitrary function of y.

However, we can absorb this function into our metric co-ordinate dy since that is the only place g appears, so $gh(y)dy \to gdy'$. Requiring that the metric is asymptotically Minkowski means we can set $f^2 g = 1$ and therefore

$$
f^2 = g^{-1}.
$$

This reproduces, for example, the constraint $\partial_{\bar{N}} \ln g = -2\partial_{\bar{N}} \ln f$ labelled as equation (8) in [31]. Both the components $F_{01y\beta}$ and its Hodge dual seven-form components $F_{23MNPQR}$ (where $MNPQR$ are a non-trivial combination of holomorphic and anti-holomorphic indices) then vanish. This also implies that the contraction

$$
\iota_\Omega F = 0, \tag{3.117}
$$

which will simplify the calculations in what follows. We shall proceed in a similar manner in the analysis of the other differential equations.

The differential equation for Σ (3.116) yields numerous results. Foremost among them are the non-trivial components of the field strength F

$$F_{MN\bar{P}\bar{Q}} = -\partial_y \left[f^2 \left(G_{N\bar{P}} G_{M\bar{Q}} - G_{N\bar{Q}} G_{M\bar{P}} \right) \right] \tag{3.118}$$

$$F_{PQ\bar{R}y} = \partial_P \left(f^{-2} G_{Q\bar{R}} \right) - \partial_Q \left(f^{-2} G_{P\bar{Q}} \right) \tag{3.119}$$

and their complex conjugates. The second result is calculated from the Hodge dual seven-form components $F_{0123M\bar{N}S}$, where we have used the conventions outlined in Appendix A.

There are also relations between the undetermined functions f, g and the determinant of the Hermitian part of the metric G. Concretely, defining a new function H, we have

$$H^2 |P(z)|^2 \equiv g^6 |P(z)|^2 = |\text{det G}| \tag{3.120}$$

with P an arbitrary holomorphic function of z^M. This allows for the freedom to make a holomorphic change of variables, in agreement with the observations of [31,82,83]. In our co-ordinates we have chosen $P(z) = 1$.

Furthermore, if we rescale the metric such that

$$g_{M\bar{N}} = H^{-1/6} G_{M\bar{N}} \tag{3.121}$$

its associated Hermitian 2-form becomes

$$\omega = i g_{M\bar{N}} dz^M \wedge dz^{\bar{N}}. \tag{3.122}$$

In this form a further constraint derived from our differential equation can be succinctly written as in (2.40)

$$\partial \left(\bar{\partial} \right) (\omega \wedge \omega) = \partial * \omega = 0. \tag{3.123}$$

This co-Kähler calibration agrees with the constraints on generalised calibrations typical of these spacetimes [84]. We shall use this constraint to calculate the Kähler metric on the moduli space of complex scalars of the $\mathcal{N}=1$ gauge theory in the next subsection.

One can check that the rest of the constraints listed as (6-13) in [31] are reproduced in their entirety. In addition, one must check that the equations of motion for the four-form field strength and the Bianchi identity are satisfied.

Since we are considering an M5-brane geometry, which couples magnetically to the three-form potential, the roles of the Bianchi identity and the equation of motion are reversed. This means we require that $d*F=0$ trivially. This is satisfied with the non-trivial components of F we have calculated.

In summary, in terms of the rescaled metric, the solution is in agreement with [31] (as in (2.38), reproduced here for convenience):

$$\begin{aligned} ds^2 &= H^{-1/3}dx^2{}_{(1,3)} + 2H^{1/6}g_{M\bar{N}}dz^M dz^{\bar{N}} + H^{2/3}dy^2 && (3.124)\\ F &= \partial_y(\omega\wedge\omega) - i\partial(H^{1/2}\omega)\wedge dy + i\bar{\partial}(H^{1/2}\omega)\wedge dy. && (3.125) \end{aligned}$$

The equation of motion for F takes the form:

$$dF = \partial_y\left(\omega\wedge\omega\right)\wedge dy - 2i\bar{\partial}\partial\left(H^{1/2}\omega\right)\wedge dy = J, \tag{3.126}$$

where J denotes the source five-form specifying the shape of the Riemann surface describing our M5-brane configuration.

This method is quite similar in spirit to the G-structures approach developed in [24, 29, 30], and it was shown in [30] how that group theoretic approach could re-derive this same supergravity solution.

3.4.2 Probe calculation of complex scalar moduli space

We can perform a similar probe calculation to that of Section 3.1.1 to determine the form of the moduli space of the complex scalars, although now in the $\mathcal{N}=1$ supersymmetric gauge theory case. The main difference is that now the M5-brane is probing a background of M5-branes wrapping 2-cycles in $\mathbf{C}^3$ (instead of $\mathbf{C}^2$).

The metric we are using for this background, in terms of co-ordinates similar to the F^2, G co-ordinates of the $\mathcal{N}=2$ case, is

$$ds^2 = H^{-1/3}dx^2{}_{(1,3)} + 2\left(H^{-1/3}\left|dG\right|^2 + H^{2/3}\left|dF^2\right|^2 + H^{2/3}\left|dK^2\right|^2\right) + H^{2/3}dy^2 \tag{3.127}$$

where, as before, F^2 and K^2 can be thought of a local co-ordinates perpendicular to the background M5-brane, and G as locally parallel to the background M5-brane. This implies that the Jacobian with respect to the z^i $(i=1,2,3)$ must be equal to one. It must also be the case that H is harmonic in F^2, K^2 and y. One can check that this form of the metric satisfies the equations of motion. This could be used for explicit calculations but we will be able to show that $K_{\alpha\bar{\beta}}$ is Kähler from the general equations of motion (2.38,2.40).

The holomorphic embeddings are now:

$$\begin{aligned}
X^m &= \sigma^m \\
X^M &= X^M\left(z, \sigma^m, u_\alpha\left(\sigma^m\right)\right) \\
X^{\bar{N}} &= X^{\bar{N}}\left(\bar{z}, \sigma^m, u_{\bar{\beta}}\left(\sigma^m\right)\right) \\
X^y &= X^y\left(z, \bar{z}, \sigma^m, u_\alpha\left(\sigma^m\right), u_{\bar{\beta}}\left(\sigma^m\right)\right),
\end{aligned}$$

with $m = 0\ldots3$, $M, N = F^2, K^2, G$ and y refers to $x^{(10)}$, the totally transverse direction. The $z, \bar{z}$ are arbitrary complex co-ordinates on the M5-brane worldvolume. The same arguments about small deviations from a supersymmetric embedding of our probe in the X^y directions we used previously also apply in this case.

The calibration bound satisfied by our M5-brane probe is also different to the previous case where we probed a background of M5-branes wrapping a 2-cycle in $\mathbf{C}^2$. Then, it was a Kähler calibration which the probe had to satisfy. From the previous subsection, for a background of M5-branes wrapping a 2-cycle in $\mathbf{C}^3$, the calibration bound our probe has to satisfy is given by Equation (3.123). In terms of the metric $G_{M\bar{N}}$ (which we recall is $G_{M\bar{N}} = H^{1/6}g_{M\bar{N}}$), the constraint takes the form

$$\partial_{[R}\left(H^{-1/3}\left(G_{M|\bar{P}|}G_{N|\bar{Q}|} - G_{M|\bar{Q}|}G_{N]\bar{P}}\right)\right) = 0. \tag{3.128}$$

This constraint is an essential element in the calculation.

Repeating the analysis of Section 3.1.1 reveals, with the appropriate extension to $\mathbf{C}^3$, the following action for the M5-brane probe

$$S=\int d^4\sigma d^2z\; H^{-2/3}g_{z\bar{z}}\sqrt{-\det\left(\eta_{mn}+H^{1/3}L_{mn}\right)} \tag{3.129}$$

with L_{mn} of the same form as before. Expanding this action and looking at the quadratic terms in the complex moduli only, we find

$$S_{kin}=\tau_5\int d^4\sigma\;\partial_m u_\alpha\partial^m u_{\bar{\beta}}K_{\alpha\bar{\beta}}, \tag{3.130}$$

where $K_{\alpha\bar{\beta}}$ is a Kähler metric given by

$$\begin{aligned} K_{\alpha\bar{\beta}} &= \int_\Lambda d^2z\; H^{-1/3}\left(G_{\alpha\bar{\beta}}G_{z\bar{z}}-G_{\alpha\bar{z}}G_{z\bar{\beta}}\right) &(3.131)\\ &= \int_\Lambda d^2z\; T_{MN\bar{\beta}}\partial_\alpha X^M\partial_z X^N. &(3.132)\end{aligned}$$

We have introduced the notation

$$T_{MN\bar{\beta}}\equiv H^{-1/3}\left(G_{M\bar{P}}G_{N\bar{Q}}-G_{M\bar{Q}}G_{N\bar{P}}\right)\left(\overline{\partial_\beta X^P}\right)\left(\overline{\partial_z X^Q}\right). \tag{3.133}$$

As before, $G_{\alpha\bar{\beta}}\equiv\frac{\partial X^M}{\partial u_\alpha}G_{M\bar{N}}\frac{\partial X^{\bar{N}}}{\partial u_{\bar{\beta}}}$, and $G_{M\bar{N}}$ refers to the spacetime metric (3.121). In our notation, we have that $\partial_{[P}T_{MN]\bar{\beta}}=0$ (from (3.123) or (3.128)) and also that $T_{MN\bar{\beta}}=-T_{NM\bar{\beta}}$.

The form of the metric $K_{\alpha\bar{\beta}}$ is quite suggestive, and using the constraint (3.128), we can show this is Kähler up to total derivative boundary terms. Taking the anti-symmetrised derivative of this metric we get

$$\begin{aligned}\partial_{[\gamma}K_{\alpha]\bar{\beta}} = &\int_\Lambda d^2z\;\partial_\gamma\left(T_{MN\bar{\beta}}\right)\partial_\alpha X^M\partial_z X^N\\ &-\int_\Lambda d^2z\;\partial_\alpha\left(T_{MN\bar{\beta}}\right)\partial_\gamma X^M\partial_z X^N\\ &+\int_\Lambda d^2z\; T_{MN\bar{\beta}}\partial_\alpha X^M\left(\partial_\gamma\partial_z X^N\right)\\ &+\int_\Lambda d^2z\; T_{MN\bar{\beta}}\partial_\gamma X^N\left(\partial_\alpha\partial_z X^M\right).\end{aligned} \tag{3.134}$$

Integrating the third term by parts and simplifying the result we have

$$\begin{aligned}
\partial_{[\gamma}K_{\alpha]\bar{\beta}} &= \int_{\Lambda} d^2z\, \left(\partial_R T_{MN\bar{\beta}}\right) \partial_\alpha X^M \partial_z X^N \partial_\gamma X^R \\
&\quad -\int_{\Lambda} d^2z\, \left(\partial_R T_{MN\bar{\beta}}\right) \partial_\gamma X^M \partial_z X^N \partial_\alpha X^R \\
&\quad -\int_{\Lambda} d^2z\, \left(\partial_R T_{MN\bar{\beta}}\right) \partial_\alpha X^M \partial_\gamma X^N \partial_z X^R \\
&\quad +\int_{\Lambda} d^2z\, \partial_z \left(T_{MN\bar{\beta}} \partial_\alpha X^M \partial_z X^N\right) \\
&= \partial_\alpha X^M \partial_z X^N \partial_\gamma X^R \left[\partial_R T_{MN\bar{\beta}} - \partial_M T_{RN\bar{\beta}} - \partial_N T_{MR\bar{\beta}}\right] \\
&\quad +\int_{\Lambda} d^2z\, \partial_z \left(T_{MN\bar{\beta}} \partial_\alpha X^M \partial_z X^N\right) \\
&= \int_{\Lambda} d^2z\, \partial_z \left(T_{MN\bar{\beta}} \partial_\alpha X^M \partial_z X^N\right). \qquad (3.135)
\end{aligned}$$

So again we have a total derivative for the boundary term, and the moduli space metric is indeed Kähler. An important role was played by the spacetime calibration bound (3.128) (or equivalently (3.123)) in analogy with the calculation of Section 3.1.1. However, note that in this case it is not a Kähler calibration but part of the more generalised calibrations typical of these spacetimes.

3.4.3 Dimensional reduction of the background supergravity solution

This follows very similar lines to the $\mathcal{N} = 2$ case, so again we just give the results. We find the eleven-dimensional metric becomes

$$ds_{11}^2 = H^{-1/3} dx_{(1,3)}^2 + 2H^{1/6} M_{\mu\nu} dx^\mu dx^\nu + 2H^{1/6} ds_{KK}^2 + H^{2/3} dy^2 \qquad (3.136)$$

where now

$$ds_{KK}^2 = g_{w\bar{w}} \left(d\tilde{x}^7 + D_\mu dx^\mu\right)^2. \qquad (3.137)$$

If, as before, we denote the $\mu\nu = (a, b, \tilde{6})$, then the $M_{\mu\nu}$ part corresponds to

$$\begin{aligned}
M_{\mu\nu} dx^\mu dx^\nu &= g_{w\bar{w}} \cos^2\theta d\tilde{x}^6 + \cos\theta (g_{a\bar{w}} da + g_{w\bar{a}} d\bar{a}) d\tilde{x}^6 \\
&\quad + g_{a\bar{b}} da d\bar{b} - \frac{1}{4 g_{w\bar{w}}} (g_{a\bar{w}} da - g_{w\bar{a}} d\bar{a})^2 \qquad (3.138)
\end{aligned}$$

where now $a, b = v, s$ and D_μ is in general given by

$$D_\mu dx^\mu = -\frac{i}{2g_{w\bar{w}}}(g_{a\bar{w}}da + g_{w\bar{a}}d\bar{a}) + \sin\theta d\tilde{x}^6. \tag{3.139}$$

Using the same form of the Kaluza-Klein reduction ansätze (3.54) used previously, we can easily read off the various supergravity fields. In particular, the dilaton is now

$$\phi = \frac{3}{4}\ln\left(2g_{w\bar{w}}H^{1/6}\right) \tag{3.140}$$

and the six component of the R-R one form becomes

$$C_6^{(1)} = H^{-1/8}\sin\theta\left[8g_{w\bar{w}}^3\right]^{(1/4)}. \tag{3.141}$$

3.4.4 The Yang-Mills coupling and theta angle

The ten-dimensional string frame background metric is given by

$$ds_{10}^2 = \sqrt{2g_{w\bar{w}}}\left[H^{-1/4}dx_{(1,3)}^2 + 2H^{1/4}M_{\mu\nu}dx^\mu dx^\nu + H^{3/4}dy^2\right]. \tag{3.142}$$

If we now place our probe so that it lies along the 0123[$\tilde{6}$] directions, we can calculate the induced metric and thus the Yang-Mills coupling. This turns out to be

$$\frac{1}{g_{YM}^2} = \frac{N}{8\pi^2 g_s l_s}\int d\tilde{x}^6\ \cos\theta \tag{3.143}$$

where to evaluate the determinant of the induced metric we have used $M_{\tilde{6}\tilde{6}} = g_{w\bar{w}}\cos^2\theta$, which follows from the Hermitian condition of the metric components. Again we now include a factor of N to take into account the possibility of the probe wrapping the x^7 direction N times.

The theta angle turns out to be quite simple as well, explicitly

$$\Theta_{YM} = \left[\frac{N}{g_s l_s}\int d\tilde{x}^6\ (\sin\theta)\right] \quad \mathrm{mod}\ 2\pi N. \tag{3.144}$$

We note that these results are in exact agreement with the previous $\mathcal{N} = 2$ results. As before, the theta angle is constant for the same reasons: the endpoints are held at fixed points in the x^7 direction. In this case, however, there does not seem to be a supersymmetric probe brane capable of detecting the running of the Yang-Mills gauge coupling since all such branes would necessarily reside at the origin in the $|v|$-plane and be fixed there. As in the pure $\mathcal{N} = 1$ Yang-Mills case, when all N D4-branes are coincident due to the rotation of one of the NS5-branes, we again have a classical gauge coupling proportional to L_6, the separation of the NS5-branes at that particular point. The theta angle would be proportional to L_θ, the distance between the endpoints of the D4-branes in the x^7 direction.

3.4.5 Instantons

We can also probe the $\mathcal{N} = 1$ background along $\tilde{x}^6$ with Euclidean D0-brane probes to find the corresponding instanton action in the D4-brane worldvolume gauge theory. This turns out to be exactly the same as for the $\mathcal{N} = 2$ case, concretely

$$S_{D0} = \frac{8\pi^2}{g_{YM}^2} - i\Theta_{YM}, \tag{3.145}$$

which is the correct form of the instanton action.

3.5 Discussion

In this chapter we have used M-branes as probes of the supersymmetric eleven-dimensional supergravity solutions [31,46,48] corresponding to M5-branes wrapping 2-cycles in $\mathbf{C}^2$ and $\mathbf{C}^3$. These probes have revealed interesting features about the corresponding $\mathcal{N} = 2$ and $\mathcal{N} = 1$ field theories. In general there were no unwanted supergravity corrections to field theory parameters such as the gauge coupling and theta angle from this analysis.

In the case of M5-brane probes, we have determined that the Kähler metric for the kinetic term of the complex scalars in the $\mathcal{N} = 2$ effective Lagrangian receives no supergravity corrections. This is also true of the gauge coupling and theta angle parameters. The static M2-brane probe calculation, probing the BPS spectra and corresponding to a monopole in the field theory, also agrees with the usual calibration arguments.

We demonstrated a new derivation of the supergravity solution [31] using a method where the projection conditions and spinor differential equations played a central role. We also analysed the $\mathcal{N} = 1$ field theory related to M5-branes wrapping a 2-cycle in $\mathbf{C}^3$. All the results showed no supergravity corrections to the usual flat-space field theory analysis.

Chapter 4

Calibrations, central charges, structure groups and some field theory

The purpose of this chapter is to introduce and further develop some concepts which we may have already seen and which will be required for the next chapter. In particular, a further look at generalised calibrations, and their relation to central charges of probe branes and also M-theory structure groups. These three concepts are related by examining the consequences of having non-zero flux in an eleven-dimensional supergravity background. With non-trivial flux, the special holonomy manifolds we have previously introduced get a reduced group structure, since the usual connection acquires an intrinsic torsion. We shall also find that studying the general superalgebras of these backgrounds coupled to probe branes implies the existence of closed forms which may be interpreted as generalised calibration forms. Finally, we look at some possible field theory interpretations of these brane configurations, such as BPS instantons, monopoles, vortices and domain walls.

4.1 Generalised calibrations

In Chapter 2, we introduced the concept of calibrations and considered supergravity backgrounds with zero flux. We now consider eleven-dimensional solutions with non-zero four-form field strength, $F \neq 0$. These are the types of backgrounds we wish to analyse in more depth. Our supergravity solutions of M5-branes wrapped on two-cycles are of this form.

As we saw previously, the Killing spinor equation shows that when one considers a background with flux, the Killing spinors of the brane configuration in question are determined by both the metric and field strength, and thus no longer covariantly constant. This is because we must take into account the coupling of the branes to the gauge potential. If we do that, which is equivalent to saying that the energy of a brane is a measure of not just the volume but also the charge, then we find that we again get calibrated surfaces of a more general kind.

We can, however, still construct generalised calibrations from the Killing spinors of the background [38,44]. In a supersymmetric background such as the ones we are considering, this implies that the quantity minimised by a calibrated cycle is related to the energy of a brane. It is also the case that these calibrated cycles are therefore supersymmetric.

If we consider a p-dimensional submanifold $\mathcal{M}_p$ of $\mathcal{M}$ with non-trivial field strength $F = dA$, then we define general calibrations such that:

$$\begin{aligned} d(\phi_p + A_p) &= 0 \qquad (4.1) \\ \mathcal{P}_{T_x\mathcal{M}_p}\phi &= vol(T_x\mathcal{M}_p). \qquad (4.2) \end{aligned}$$

So then $\mathcal{M}_p$ is called calibrated if, as well as (4.1), at every point on $\mathcal{M}_p$, the pullback of ϕ to some tangent space $T_x\mathcal{M}_p$ is equal to the volume form on that tangent space. Note that the condition for a calibrated submanifold is a local one.

It is clear from the above that a generalised calibration ϕ_p is not closed, but rather gauge equivalent to the potential A_p. In the trivial case where this gauge field vanishes, this reduces to the standard calibrations. There are also more complicated forms for the calibration to have in general, since we have relaxed the requirement that ϕ_p is closed. We shall see more complicated generalised calibrations, involving also worldvolume fields and the Killing spinors of the background when we discuss central charges in the next section.

The statement we made about calibrating submanifolds minimising volume in their homology class now holds for generalised calibrations too, except that calibrated submanifolds can now be associated with minimum energy configurations.

If the background is supersymmetric, there is a natural construction of ϕ using Killing spinors. These are Killing spinors of the background with non-trivial flux, whose Killing spinor equation is given schematically by $\nabla\epsilon + F \cdot \Gamma\epsilon = 0$. We can now construct a p-form in the following manner:

$$\phi = \frac{1}{p!}\bar{\epsilon}\Gamma_{i_1 \ldots i_p}\epsilon dx^{i_1} \wedge \ldots \wedge dx^{i_p} \tag{4.3}$$

where $\bar{\epsilon} = \epsilon^T \Gamma^{\hat{0}}$. It follows from the antisymmetric properties of gamma matrices that the p-form will vanish unless $p = 1, 2 \mod 4$. Now, derivatives of these forms can be found by looking at the differential equations (2.18), which we recall are descended from the Killing spinor equation.

For the purposes of illustration, we can look briefly at an M2-brane probe example and show how these calibrating forms can be associated with its energy when considering a supergravity background with non-zero flux. The case of the M5-brane is analogous but more lengthy. With this is mind, the two-form ϕ_2 (which we called Ω in Chapter 3) is related to the field strength by the differential equation $d\phi_2 = \iota_K F$. For a certain gauge choice of A, one finds that $\iota_K F = -d\iota_K A$ and so $d\phi_2 = -d\iota_K A$.

We consider placing a probe M2-brane in a static supergravity background with $F = dA \neq 0$. If we specify the worldvolume coordinates by t, σ^1, σ^2, then the energy functional minimised by this brane is given by [38]

$$E = T_2 \int d^2\sigma \left(\eta\sqrt{-\gamma_{ab}} + \iota_K A\right) \tag{4.4}$$

where we have denoted the time-like Killing vector $K = \partial/\partial t$, with norm $\eta = \sqrt{-K^2}$. The γ_{ab} $a, b = 1, 2$ is the induced metric determinant on the spatial worldvolume of the brane, and the pullback of the two-form $\iota_K A$ is understood.

The first term of this expression corresponds to the volume of the brane (modulo the redshift factor η), and the second is the contribution from the electrostatic energy. Our aim is to show that this is indeed equivalent to a calibrated submanifold given by ϕ_2.

We can do this by considering a pair of two-dimensional submanifolds of our eleven-dimensional background, $\mathcal{A} = \mathcal{B}+\partial\Delta_0$, which are in the same homology class. We make the choice that submanifold $\mathcal{A}$ is calibrated by ϕ_2. From definition (4.2) we have that

$$Vol(\mathcal{A}) = \int_{x\in\mathcal{A}} vol(T_x\mathcal{A}) = \int_{\mathcal{A}} \phi_2 = \int_{\mathcal{B}+\partial\Delta_0} \phi_2 \leq Vol(\mathcal{B}) + \int_{\partial\Delta_0} \phi_2 \tag{4.5}$$

Using an auxiliary two-dimensional submanifold $\mathcal{C} = \mathcal{A} - \partial\Delta_1 = \mathcal{B} - \partial\Delta_2$ we can relate the boundary terms such that $\partial\Delta_0 = \partial\Delta_1 - \partial\Delta_2$. The inequality then becomes

$$Vol(\mathcal{A}) - \int_{\Delta_1} d\phi_2 \;\leq\; Vol(\mathcal{B}) - \int_{\Delta_2} d\phi_2, \tag{4.6}$$

where we have used Stoke's theorem. Therefore, we have that the calibrated submanifold $\mathcal{A}$ minimises the quantity

$$Vol(\mathcal{A}) - \int_{\Delta_1} d\phi_2 \tag{4.7}$$

in its homology class. Comparing this to the second term of (4.4) we see that provided we make the identification $d\phi_2 = -d\iota_K A$ (which on the other hand is a consequence of the supersymmetry of the background and our spinor construction), we have

$$Vol(\mathcal{A}) - \int_{\Delta_1} d\phi_2 = Vol(\mathcal{A}) + \int_{\partial\Delta_1} \iota_K A = Vol(\mathcal{A}) + \int_{\mathcal{A}} \iota_K A \tag{4.8}$$

We have ignored the constant term $\int_{\mathcal{C}} \iota_K A$ which cancels from both sides of (4.6). Since the volume form can be identified with the induced metric determinant of the spatial worldvolume of the M2-brane, we have shown that the calibrating form ϕ_2 constructed from Killing spinors of the background indeed corresponds to minimal energy submanifolds for branes to wrap. Furthermore, branes wrapping these calibrated cycles are automatically supersymmetric. We shall look at more general examples, such as the M5-brane which includes worldvolume fields as well as the background six-form potential in the following section.

4.2 Topological charges for probe branes

As we shall see, possible supersymmetric brane probes allowed by the background geometry are determined by the central charges of the brane configuration in question. In general these charges, being topological in nature, provide a clear picture of allowed supersymmetric objects in that particular background, and possible field theory interpretations.

The bilinear spinor construction will be very useful to elucidate the relation between the calibrating forms of the backgrounds under consideration. We recall that the one-, two- and five-forms we can build from the Killing spinors satisfy a set of algebraic relation which follow from Fierz identities. They also form part of $Spin(1,10)$, and some properties of this group are useful in classifying these relations.

In particular, if we consider the Lorentz scalar built from the Killing vector $K^2 = K_\mu K^\mu$, we can ask what the possible orbits of this scalar are under $Spin(1,10)$ [85]. Since K^2 remains fixed under the orbits of the group, we may label these orbits by the value of K^2. It turns out there are only two possibilities, namely $K^2 = 0$ or $K^2 < 0$. Any spinors with $K^2 < 0$ can always be rescaled and related by a Lorentz transformation. Considering these two cases gives an efficient way to define identities between the forms K, Ω and Σ.

If we consider the time-like case with $K^2 < 0$, a possible set of projection conditions which define the spinor ϵ (up to a scale) is [24]

$$\hat{\Gamma}_{012}\epsilon = \hat{\Gamma}_{034}\epsilon = \hat{\Gamma}_{056}\epsilon = \hat{\Gamma}_{078}\epsilon = \hat{\Gamma}_{09(10)}\epsilon = \epsilon \tag{4.9}$$

$$\hat{\Gamma}_{013579}\epsilon = \epsilon \tag{4.10}$$

where we have denoted the tangent space gamma matrices by a hat. Using the identity $\hat{\Gamma}_{0123456789(10)} \equiv 1$ we see that one of the six conditions above is already implied so there are in fact only five independent, commuting projections.

In this framework, the forms K, Ω and Σ can be expressed in the following way:

$$\begin{aligned} K &= \Delta e^0 && (4.11)\\ \Omega &= \Delta(e^1\wedge e^2 + e^3\wedge e^4 + e^5\wedge e^6 + e^7\wedge e^8 + e^9\wedge e^{(10)}) && (4.12)\\ \Sigma &= \frac{1}{2}\Delta^{-2}K\wedge\Omega\wedge\Omega + \Delta Re(\Omega_5) && (4.13)\end{aligned}$$

where Ω_5 is the holomorphic five-form

$$\Omega_5 = (e^1 + ie^2)\wedge(e^3 + ie^4)\wedge(e^5 + ie^6)\wedge(e^7 + ie^8)\wedge(e^9 + ie^{(10)}). \qquad (4.14)$$

We have normalised the scale of the Killing spinor by fixing $\epsilon^T\epsilon = \Delta$. It is made explicit that K is indeed a time-like vector and that the forms define an $SU(5)$ structure on the underlying manifold. This actually corresponds to the stability group of ϵ and is also called the G-structure. In Ref. [24], the authors looked at the problem of using this $SU(5)$ structure corresponding to a time-like vector K to classify supersymmetric supergravity solutions and determine, to a large extent, properties of the metric and four-form field strength.

We now consider the case where $K^2 = 0$. The stability group of the spinor ϵ defines a $(Spin(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$ structure. The projection conditions for any Killing spinor with null K (up to an appropriate choice of vielbein) are:

$$\begin{aligned} \hat{\Gamma}_{01}\epsilon &= \epsilon\\ \hat{\Gamma}_{2345}\epsilon = \hat{\Gamma}_{2367}\epsilon &= \hat{\Gamma}_{2389}\epsilon = \hat{\Gamma}_{2468}\epsilon = -\epsilon \qquad (4.15)\end{aligned}$$

Combining these conditions with the identity $\hat{\Gamma}_{0123456789(10)} \equiv 1$ we find the additional equation

$$\hat{\Gamma}_{(10)}\epsilon = -\epsilon. \qquad (4.16)$$

With this group structure, the (K,Ω,Σ) forms are given by

$$\begin{aligned} K &= \Delta(e^0 + e^1)\\ \Omega &= -K\wedge e^{(10)}\\ \Sigma &= -K\wedge\Phi_{(4)} \qquad (4.17)\end{aligned}$$

where $\Phi_{(4)}$ is the Cayley four-form,

$$\begin{aligned}\Phi_{(4)} \;=\;& e^{2345}+e^{6789}+e^{2367}-e^{2569}-e^{3478}+e^{2468}+e^{3579}\\ & e^{4589}+e^{4567}-e^{3469}+e^{2389}-e^{2578}-e^{2479}-e^{3568}\end{aligned} \tag{4.18}$$

with the notation $e^{2345}=e^2\wedge e^3\wedge e^4\wedge e^5$. As we shall see in the next chapter, and which is already glimpsed by the above equations, the above projections conditions are reminiscent of those satisfied by an M5-brane wrapped on a holomorphic curve. In the example of the time-like K, we find this is naturally suited to describe the M5-brane wrapped on a 2-cycle in $\mathbf{C}^2$. Some of the projection conditions, with an appropriate choice of vielbein, will correspond to certain BPS states of the field theory (once we have reduced to ten-dimensional Type IIA $\mathcal{N}=2$ Hanany-Witten models) such as monopoles and vortices, for example.

Likewise, for the null K case, this set of projection conditions are naturally suited to describe the $\mathcal{N}=1$ solution of M5-branes wrapped on 2-cycles in $\mathbf{C}^3$. Various BPS states of the corresponding MQCD gauge theory are identified with certain calibrating forms, such as domain walls and Cayley four-forms, for example. We shall see this in more detail in the next chapter.

Supersymmetry algebras

We consider the reduced part of the supersymmetry algebra which involves the anti-commutator of the 32-component Majorana spinors Q_α. In eleven-dimensional flat Minkoswki space, this is simply

$$\{Q_\alpha, Q_\beta\} = (C\Gamma_m)P^m \tag{4.19}$$

where we have denoted the conjugation matrix by C. We can take this to be $\hat{\Gamma}_0$ from now on. The P^m are the translation generators.

We may introduce a constant commuting Majorana spinor ϵ^α such that the anti-commutator becomes

$$\{\epsilon^\alpha Q_\alpha, \epsilon^\beta Q_\beta\} = (\epsilon^T\hat{\Gamma}_0\Gamma_m\epsilon)P^m \tag{4.20}$$

The idea is that now the term in the round brackets is looking like our expression for the Killing vector $K_m = \bar{\epsilon}\hat{\Gamma}_m\epsilon$, since for Majorana spinors $e^T\hat{\Gamma}_0 = \bar{\epsilon}$. This we are allowed to do since the set of all constant spinors in flat space is the set of Killing spinors. The construction using the Killing spinors implies the components of K are constant since ϵ is constant.

This idea may be extended to more general supersymmetric backgrounds with curvature. In that case, as we have discussed, the backgrounds will in general possess at least one Killing spinor (or more depending on the amount of preserved supersymmetry). They will not be constant however, but rather depend on the spacetime co-ordinates. Any such Killing spinor $\epsilon^\alpha(x)$ will therefore have a corresponding supercharge ϵQ. The dependence of ϵ on the spacetime co-ordinates also implies that K is now a field.

Using the short-hand notation

$$2(\epsilon Q)^2 = K_M P^M \tag{4.21}$$

we expect $K_M P^M$ to be bosonic symmetries of the supersymmetric solution. These are associated with a vector field acting by the Lie derivative. In this case, the vector field $K(x)$ will act on the supergravity fields by $\mathcal{L}_K$. For a supersymmetric solution of eleven-dimensional supergravity, bosonic symmetries are associated with vector fields $K_M(x)$ which satisfy the Killing equations

$$\mathcal{L}_K g = 0 \tag{4.22}$$

and

$$\mathcal{L}_K F = 0 \tag{4.23}$$

with (g, F) denoting the metric and four-form field strength. This is actually an automatic consequence of the Killing spinor equations. To prove this one uses the identity

$$\mathcal{L}_X \alpha = d(\iota_X \alpha) + \iota_X d\alpha \tag{4.24}$$

where X is a vector and α a p-form. Applying this to (3.115), namely

$$d\Omega = \iota_K F \tag{4.25}$$

directly leads to the result $\mathcal{L}_K F = 0$.

There may, in general, be other isometries that are not generated by (4.21) since not all Killing vectors of a background may be constructed from the Killing spinors. In general, the supersymmetry algebra for a background is determined by the Killing spinors only up to purely bosonic factors.

Supersymmetry algebra with central charges

We may consider the possibility of central charges in the supersymmetry algebra [86–88]. The origin of the central charge is easy to understand: The supersymmetry charges Q are space integrals of local expressions in the fields (in particular the time component of the super-currents). In calculating the anti-commutator, one encounters surface terms which are normally neglected. However, in the presence of electric and magnetic charges, these surface terms are non-zero and give rise to a central charge.

Since in a general eleven-dimensional background there is both an electric three-form potential and its Hodge dual magnetic six-form potential, we have the possibility of coupling both a probe M2-brane as well as a probe M5-brane to our flat background. If we examine the simpler case of an M2-brane first, we find that this induces a modification to the anti-commutator of the fermionic charges such that

$$2(\epsilon Q)^2 = K_M P^M + \frac{1}{2}\Omega_{MN} Z^{MN} \tag{4.26}$$

where Ω_{MN} are components of the spinor bilinear two-form we have previously constructed from Killing spinors (2.18), and Z^{MN} is given explicitly by

$$Z^{MN} = \pm \int \epsilon^{ij} \frac{\partial x^M}{\partial \sigma^i} \frac{\partial x^N}{\partial \sigma^j} d^2\sigma \tag{4.27}$$

and the integral is over the spatial worldvolume of the M2-brane with co-ordinates (σ^1, σ^2). The $\pm$ refer to a brane/anti-brane. If we rewrite the momentum, P^M, as an integral of the momentum density $p^M(\sigma)$ over the spatial worldvolume of the brane, we obtain

$$2(\epsilon Q)^2 = \int d^2\sigma\, K_M p^M(\sigma) \pm \int \Omega \tag{4.28}$$

with the second term combining to give the integral of the two-form Ω. Considering more general supersymmetric backgrounds with non-zero flux, it turns out that the appropriate generalisation to the super-translation algebra is given by

$$2(\epsilon Q)^2 = \int d^2\sigma\, K_M p^M(\sigma) \mp \int (\Omega + \imath_K A)\,. \tag{4.29}$$

where $F = dA$ is the three-form gauge supergravity gauge potential. This expression is valid for general K. The term $(\Omega + \imath_K A)$ is closed and therefore of a topological nature. This combination seems quite natural and is analogous to the replacement of p_M with $p_M + A_M$ for a particle in an electromagnetic field.

Since $2(\epsilon Q)^2 \geq 0$, this gives rise to a BPS bound on the energy/momentum of the brane

$$\int d^2\sigma\, K_M p^M(\sigma) \geq \mp \int (\Omega + \imath_K A)\,. \tag{4.30}$$

where the term on the right hand side is topological. We note it is only defined up to addition of a closed one-from.

The connection between this bound and generalised calibrations can be made explicit. In particular, if we assume the background possesses a time-like Killing spinor K, then the LHS of the bound becomes $-p_0 = \mathcal{H}$, with $\mathcal{H}$ the Hamiltonian density. Now, we saw earlier (4.4) that this is given by $\mathcal{H} = Vol + \imath_K A$. Therefore the bound simply reduces to $\int d^2\sigma \geq \mp \int \Omega$ as expected. Also, from the differential equation for the forms $d\Omega = \iota_K F$, and choosing the gauge $\mathcal{L}_K A = 0$, then $\iota_K F = -d(\iota_K A)$ and so $d\Omega = -d(\iota_K A)$, as required for generalised calibrations.

We may also consider the case of coupling a probe M5-brane to a flat supergravity background. The story is analogous to that of the M2-brane. The supersymmetry algebra acquires a central charge. One may then consider the general situation of non-trivial worldvolume fields and fluxes.

The spacetime superalgebra that results is the general eleven-dimensional super-Poincaré algebra coupled to a five-brane and non-zero background flux and worldvolume fields [77]. In its most general form, valid for either time-like or null K, the supersymmetry algebra on the worldvolume of a probe M5-brane becomes:

$$2\,(\epsilon Q)^2 = \int K_M P^M \pm \int \left(\imath_K C + \Sigma + (A + dB)\wedge(\Omega + \imath_K A) - \frac{1}{2}A \wedge \imath_K A\right). \tag{4.31}$$

We have denoted by C the background six-form potential and A denotes the electric three-form potential, related by $dC = *dA + \frac{1}{2}A \wedge dA$. We also include the non-zero worldvolume two-form gauge field B. Since $(\epsilon Q)^2 \geq 0$, this leads to a BPS type bound on the energy/momentum of the M5-brane,

$$\int K_M P^M \geq \mp \int \left(\imath_K C + \Sigma + (A + dB)\wedge(\Omega + \imath_K A) - \frac{1}{2}A \wedge \imath_K A\right). \tag{4.32}$$

From properties of our construction, such as the fact that K is Killing and $\mathcal{L}_K F = 0$, one can check that the right hand side of the inequality is the integral of a closed form and thus represents a topological charge. The existence of such a closed form also provides examples of generalised calibration forms for arbitrary supersymmetric backgrounds. We shall look at examples of central charges of M5-brane probes in wrapped M5-brane backgrounds in the next chapter. Various BPS states are found, with interesting field theory interpretations.

4.3 M-theory structure groups

We have discussed, in previous chapters, how using the differential forms in the bilinear spinor formalism, one can classify the local form of general bosonic supersymmetric configurations of eleven-dimensional supergravity. Depending on whether the Killing vector constructed from the Killing spinor of the background is either time-like or null, these geometries display an $SU(5)$ or $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, respectively.

One drawback of the original approach to G-structures was that it could only be used for the classification of spacetimes with minimal supersymmetry. A natural question to ask is: what happens when there is more than just one Killing spinor present? What are the structure groups that arise as subgroups of the original isotropy group? A refinement of the original method has been successful in classifying these subgroups, completely in the case of null structure groups [45], and partly for time-like structure groups [89].

We recall what we mean by structure groups and their relation to the special holonomy manifolds we have discussed earlier. As we have seen, holonomy groups help organise M-theory vacua with zero flux [90]. In particular, these special holonomy manifolds arise from the existence of spinors ϵ which are parallel with respect to the spin connection

$$\nabla_I = \partial_I - \frac{1}{4}\omega_I^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}} = 0 \tag{4.33}$$

where we have adorned tangent space indices with a hat.

If we look at the case of non-zero flux, that is $F \neq 0$, we find the connection is altered and takes values in the Clifford algebra, since it depends on terms containing antisymmetrised products of Γ matrices. In particular, the spinor is now covariantly constant with respect to the connection

$$\tilde{\nabla}_I\epsilon = \nabla_I\epsilon + \theta_I(F)\cdot\epsilon = 0 \tag{4.34}$$

where $\theta_I(F)$ is the flux dependent part of the connection:

$$\theta_I(F) \equiv \frac{1}{288}F_{JKLM}\left[\Gamma_I^{\ JKLM} - 8\delta_I^J\Gamma^{KLM}\right]\epsilon = 0. \tag{4.35}$$

Therefore, when fluxes are turned on, there is deviation from holonomy which is measured by the intrinsic torsion of the connection. There still remains, however, a reduced group structure. As we saw, considering backgrounds with one Killing spinor, this can be either $SU(5)$ or $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$. The problem we are interested in is the classification of possible subgroups of these structure groups, which would correspond to geometries with more Killing spinors and therefore more supersymmetries. We wish to find the structure groups of backgrounds corresponding to our wrapped M5-brane solutions which preserve 8 and 16 real supersymmetries.

To start with we may examine the holonomy groups which are possible for M-theory vacua [90]. For static vacua, in which the Killing vector K is time-like, the holonomy group H is a subgroup of $SU(5)$. These are automatically Ricci-flat and satisfy the supergravity equations of motion. Such spacetimes are locally isometric to a product $M = \mathbf{R} \times X$ with metric

$$ds^2 = -dt^2 + ds^2(X) \tag{4.36}$$

where X is a Calabi-Yau 5-fold. Assuming the manifold is simply connected, the possible holonomy groups and corresponding number of preserved supersymmetries $N = 32\nu$ is given by Table (4.1). The notation refers to the product

$$M = \mathbf{M}^{11-d} \times W_d \tag{4.37}$$

where $\mathbf{M}^{11-d}$ is $(11-d)$-dimensional Minkowski space and d is the dimension of W. The holonomy groups $H \subset Spin(d)$ of W and the fraction ν of supersymmetry that such a geometry preserves is also specified.

d	$H \subset Spin(d)$	ν
10	$SU(5)$	$\frac{1}{32}$
10	$SU(2) \times SU(3)$	$\frac{1}{8}$
8	$Spin(7)$	$\frac{1}{16}$
8	$SU(4)$	$\frac{1}{8}$
8	$Sp(2)$	$\frac{3}{16}$
8	$Sp(1) \times Sp(1)$	$\frac{1}{4}$
7	G_2	$\frac{1}{8}$
6	$SU(3)$	$\frac{1}{4}$
4	$SU(2) \cong Sp(1)$	$\frac{1}{2}$
0	$\{1\}$	1

Table 4.1: Holonomy groups H in relation to fraction ν of preserved supersymmetry

For non-static vacua, such that K is null, then the holonomy group is a subgroup of the isotropy group $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$. The most general local metric with this holonomy group is

$$ds^2 = 2dx^+dx^- + a(dx^+)^2 + (dx^9)^2 + g_{ij}dx^i dx^j \tag{4.38}$$

where i, j run from 1 to 8, $\partial_- a = 0$ and g_{ij} is an x^+-dependent family of metrics with holonomy contained in $Spin(7)$ and obeying the property that

$$\partial_+ \Psi_{(4)} = \lambda \Psi_{(4)} + \Xi \tag{4.39}$$

where $\Psi_{(4)}$ is the self-dual $Spin(7)$-invariant Cayley four-form, $\lambda(x^+, x^-)$ a smooth function and Ξ an anti-self-dual four-form.

The holonomy groups of non-static vacua will not necessarily decompose the spacetime into a metric product. In particular, the possible subgroups are given in Table (4.2).

$H \subset Spin(1,10)$	ν
$(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$	$\frac{1}{32}$
$(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$	$\frac{1}{16}$
$(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$	$\frac{3}{32}$
$(Sp(1) \ltimes \mathbb{R}^4) \times (Sp(1) \ltimes \mathbb{R}^4) \times \mathbb{R}$	$\frac{1}{8}$
$(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$	$\frac{1}{16}$
$(SU(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$	$\frac{1}{8}$
$(Sp(1) \ltimes \mathbb{R}^8) \times \mathbb{R}$	$\frac{1}{4}$
$\mathbb{R}^9$	$\frac{1}{2}$

Table 4.2: Holonomy groups H in relation to fraction ν of preserved supersymmetry

The holonomy groups in the table are all of the form

$$(W \ltimes \mathbb{R}^d) \times \mathbb{R}^{9-d} \tag{4.40}$$

where $W \subset Spin(d)$.

When we take into account the inclusion of flux in our supergravity backgrounds, many more possible subgroups are possible, with varying numbers of preserved Killing spinors. As we mentioned, for backgrounds with more than one

Killing spinor, the refined G-structure classification is more efficient in calculations [45, 89, 91]. We shall not delve into the details here, but merely provide a brief synopsis of the method.

In particular we discuss this for the case of classifying null structure groups [45] (those subgroups which arise from backgrounds preserving a single null Killing spinor in eleven dimensions). This involves discarding the bilinear spinor construction and working directly with the spinors themselves. One finds the isotropy group of two or more independent Killing spinors, and by choosing a convenient basis for spinor space, calculates the conditions for supersymmetry on the spin connection, fluxes and intrinsic torsion.

If one takes a fiducial spinor ϵ and acts on it, in an appropriate subset of the Clifford algebra, with matrices Q, then one may span the basis of spinors. We can then assume the fiducial spinor ϵ is Killing. It follows that we can then calculate the constraints from this Killing spinor using the usual G-structures approach. Then, since $Q\epsilon$ spans the basis of spinors, we may write any other Killing spinor in the form $\epsilon_i = Q\epsilon$. The condition for ϵ_i to be Killing is then

$$[\tilde{\nabla}_I, Q]\epsilon = 0 \tag{4.41}$$

One may write the spinor $\epsilon_K = [\tilde{\nabla}_I, Q]\epsilon$ as a manifest sum of basis spinors by imposing the defining projection conditions satisfied by ϵ. Consequently, by linear independence, the coefficient of each must vanish independently.

This approach is rather different in that it does not classify configurations according to the number of preserved supersymmetries. Instead, the focus is much more on the structure groups. Starting from the assumption of the existence of one Killing spinor, the incorporation of additional Killing spinors can have one of either two effects: either a further global reduction of the structure group or more restrictions on the intrinsic torsion of the existing G-structure.

As an illustrative example we can look at the case at hand, the classification of structure groups in eleven dimensions assuming the existence of a null Killing spinor. Then, in the spacetime basis

$$ds^2 = 2e^+e^- + e^ie^i + e^9e^9 \tag{4.42}$$

where $i = 1\ldots 8$ denotes the base space manifold, we define a basis of spinors by the projections (with no sum on i)

$$\begin{aligned}
\Gamma_{1234}\epsilon_{(i)} &= -\alpha^1_{(i)}\epsilon_{(i)} \\
\Gamma_{3456}\epsilon_{(i)} &= -\alpha^2_{(i)}\epsilon_{(i)} \\
\Gamma_{5678}\epsilon_{(i)} &= -\alpha^3_{(i)}\epsilon_{(i)} \\
\Gamma_{1357}\epsilon_{(i)} &= -\alpha^4_{(i)}\epsilon_{(i)} \\
\Gamma^{\alpha^5_{(i)}}\epsilon_{(i)} &= 0
\end{aligned} \tag{4.43}$$

for the thirty-two possible combinations $\alpha^{1,\ldots,5}_{(i)} = \pm 1$. The fiducial spinor in this case can be defined by $\epsilon \equiv \epsilon_{(1)}$ so that $\alpha^{1,\ldots,5}_{(i)} = +1$. The idea is then to construct this basis by solving the projections (4.43) for the cases $\alpha^5_{(i)} = \pm$. If we consider $\alpha^5_{(i)} = +$ and chirality $\alpha^1_{(i)}\alpha^3_{(i)} = +$ we find the eight basis spinors are of the form

$$\epsilon, \quad J^A_{ij}\Gamma^{ij}\epsilon \tag{4.44}$$

where $A = 1, \ldots, 7$ and the $\mathbf{8}_+$ of $Spin(8)$ has been broken down into the $\mathbf{1} + \mathbf{7}$ of $Spin(7)$. After considering all the possible combinations from the chosen subset of the Clifford algebra, the general form of an arbitrary Majorana spinor η in eleven dimensions may be written as

$$\eta = (f + \frac{1}{8}f^A J^A_{ij}\Gamma^{ij} + u_i\Gamma^i + g\Gamma^- + \frac{1}{8}g^A J^A_{ij}\Gamma^{-ij} + v_i\Gamma^{-i})\epsilon \tag{4.45}$$

for the thirty-two real functions f, f^A, u_i, g, g^A, v_i. This clearly shows that the space of Majorana spinors in eleven dimensions is isomorphic to the direct sum of the spaces of $Spin(7)$ forms on the base space, namely

$$\Lambda^0_1 \oplus \Lambda^0_1 \oplus \Lambda^1_8 \oplus \Lambda^1_8 \oplus \Lambda^2_7 \oplus \Lambda^2_7. \tag{4.46}$$

One then assumes the fiducial spinor ϵ is Killing and considers the subgroups spanned by the different parts outlined in (4.45), from simplest to most general, such that all the possibilities are considered.

In summary, we list the structure groups which interest us. In particular, those that are always null in eleven dimensions and the number N of Killing spinors they can contain, in Table (4.3).

G	N
$*\quad (Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$	1
$*\quad (SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$	$2, 3, 4$
$(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$	$2, 3, \ldots, 6$
$(Sp(1) \ltimes \mathbb{R}^4) \times (Sp(1) \ltimes \mathbb{R}^4) \times \mathbb{R}$	$3, 4, \ldots, 8$
$(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$	5
$(U(1) \ltimes \mathbb{R}^8) \times \mathbb{R}$	6
Chiral $\mathbb{R}^9$	$7, 8$
$(G_2 \ltimes \mathbb{R}^7) \times \mathbb{R}^2$	2
$*\quad (SU(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$	$2, 3, 4$
$(SU(2) \ltimes \mathbb{R}^4) \times \mathbb{R}^5$	$3, 4, \ldots, 8$
$\mathbb{R}^9$	$3, 4, \ldots, 16$
$Spin(7)$	
G_2	

Table 4.3: G-structures which are always null in eleven dimensions

We have denoted with a $*$ those structure groups which will play a role when considering the M5-brane wrapped on a 2-cycle in $\mathbf{C}^3$ background.

4.4 Summary of field theory results

In this section we present a very brief review of the field theory results [92] which are of relevance to the central charge calculation in the next chapter. These provide field theoretic support for our group theoretic results. We examine four main objects in decreasing order of co-dimension: instantons, monopoles, vortices and domain walls, and give their brane construction as well.

Instantons in supersymmetric gauge theory

Since their discovery, Yang-Mills instantons [93] have lead to many new insights, such as their interpretation as quantum mechanical tunnelling events between inequivalent vacua. They also play a role in the analysis of the dualities relating different supersymmetric field theories [94, 95]. They are co-dimension 4 objects which in four-dimensional gauge theories are interpreted as zero-sized particles existing at a particular point in time.

To start with, we consider the bosonic part of an $SU(N)$ supersymmetric Yang-Mills theory action

$$S = \frac{1}{2g_{YM}^2} \int dx^4 \, \mathrm{Tr} F_{MN} F^{MN} \tag{4.47}$$

and look for finite action solutions to the Euclidean equations of motion

$$\mathcal{D}_M F^{MN} = 0. \tag{4.48}$$

As we move towards the the boundary $r \to \infty$ of spatial $\mathbf{R}^4$, the potential A must become a pure gauge term

$$A_M \to i g^{-1} \partial_M g \tag{4.49}$$

with $g \in SU(N)$. A finite action configuration therefore provides a map from the boundary at spatial infinity $\partial \mathbf{R}^4 \cong \mathbf{S}^3_\infty$ to the group $SU(N)$. These configurations are topologically stable and their charge $k \in \mathbf{Z}$ is known as the Pontryagin number, or second Chern class, of this map.

We may solve (4.48) and find the Bogomoln'yi bound [17] in a topological sector k. It turns out that the action for an instanton is bounded by

$$S_{inst} \geq \frac{8\pi^2}{g_{YM}^2} |k| \tag{4.50}$$

with equality if and only if F is (anti)self-dual;

$$F_{MN} = \pm \tilde{F}_{MN}. \tag{4.51}$$

We have defined the dual field strength as $\tilde{F}_{MN} = \frac{1}{2}\epsilon_{MNPQ}F^{PQ}$. We can see clearly that a solution to the self-duality equations must necessarily solve the full equations of motion since we have

$$\mathcal{D}_M F^{MN} = \mathcal{D}_M \tilde{F}^{MN} = 0 \tag{4.52}$$

by the Bianchi identity.

We may also add a so-called theta term to the action (4.47)

$$\mathcal{L}_{\Theta_{YM}} = \frac{\Theta_{YM}}{32\pi^2} F^A_{MN} \tilde{F}^{AMN} \tag{4.53}$$

which has the effect of shifting the electric charge, the Witten effect [96]. This changes the action of the instaton, which becomes

$$S_{inst} = \frac{8\pi^2}{g^2_{YM}}|k| - i\Theta_{YM}. \tag{4.54}$$

We see that this is the same result as what we got in the M5-brane probe calculations of Chapter 3, specifically Equations (3.96) and (3.145) for the value $|k| = 1$.

Brane construction of instantons

As we have already mentioned previously, D0-branes can dissolve into gauge theory instantons on the worldvolume of the D4-branes in the Hanany-Witten picture [80, 81]. More precisely, they are described by Euclidean D0-branes extended in the x^6 direction and point-like in the time dimension. This can be shown explicitly in string theory following the ADHM construction [97] for Dp-D$(p-4)$-brane systems. The existence of the D0-brane is possible since it is charged under the one-form A_1 which couples to the D4-brane worldvolume fields via the WZ term $A_1 \wedge F \wedge F$. This D0-brane will have an electric coupling to A_1 proportional to its instanton charge, as it should, and furthermore the theta angle of the gauge theory is associated with the Wilson line of A_1 along x^6. As we shall see, when we calculate the action for D0-brane probes of supergravity solutions which correspond to these type of Hanany-Witten models, we get exactly (4.54).

Monopoles in supersymmetric gauge theory

Monopoles are co-dimension 3 objects which are postulated to have a long range, radial magnetic field. They naturally occur in non-Abelian gauge theories and have many applications, such as in the Seiberg-Witten analysis of $\mathcal{N} = 2$ gauge theories [50] or S-duality of four-dimensional $\mathcal{N} = 4$ Yang-Mills [98].

A simple action with gauge group $SU(N)$ that supports the existence of monopoles is

$$S = \frac{1}{g_{YM}^2} \int dx^4 \, \mathrm{Tr} \left(\frac{1}{2} F_{MN} F^{MN} + (\mathcal{D}_M \phi)^2 \right) \tag{4.55}$$

where we have introduced the real scalar field ϕ. We may treat this as a part of the $\mathcal{N} = 2$ Lagrangian. Since there is no potential term, we may set the vev of the scalar arbitrarily to $\langle\phi\rangle = \mathrm{diag}(\phi_1, \ldots, \phi_N) = \vec{\phi} \cdot \vec{H}$ with $\vec{H}$ a basis of the Cartan subalgebra of $su(N)$ and with the constraint $\sum_{a=1}^{N} \phi_a = 0$.

Monopoles are topologically supported by the twisting of the vev $\langle\phi\rangle$ along its gauge orbit as we go around S^2, the spatial boundary at infinity. If we parametrise this space with θ and φ then we may write our soliton configurations as $\langle\phi\rangle = \langle\phi(\theta, \varphi)\rangle$. The winding maps suggest the existence of monopoles carrying magnetic charge in each of the $(N - 1)$ $U(1)$'s which are left unbroken by the vev $\langle\phi\rangle$.

To explain how the winding is associated with the magnetic charge one may consider that for finite energy configurations, $\langle\phi\rangle$ varying asymptotically means that $\partial\phi \sim 1/r$. To cancel the corresponding infrared divergence we need to set the gauge potential to $A_\theta \sim 1/r$ which implies a magnetic field of the form $B \sim 1/r^2$. The exact non-Abelian magnetic force carried by the soliton is of the form

$$B_i = \vec{g} \cdot \vec{H}(\theta, \varphi) \frac{\hat{r}_i}{4\pi r^2} \tag{4.56}$$

where $\vec{g}$ is the magnetic charge vector. The twists of the unbroken Cartan subalgebra within the $su(N)$ Lie algebra as we move around spatial infinity is suggested by the notation $\vec{H}(\theta, \varphi)$.

For example, in the singular gauge where $\langle\phi\rangle$ is fixed to be constant at infinity, the magnetic field has a diagonal form

$$B_i = \text{diag}(g_1, \ldots, g_N)\frac{\hat{r}_i}{4\pi r^2} \tag{4.57}$$

where $\sum_{a=1}^{N} g_a = 0$ and $g_a \in 2\pi\mathbf{Z}$. We find this re-introduces a Dirac string-like singularity for any single-valued gauge potential, like in the Wu and Yang gauge bundle construction [99].

Upon solving the monopole equations using the Bogomoln'yi bound, one finds that the mass is equal to a topological charge given by

$$M\text{mono} \geq \frac{2\pi}{g_{YM}^2}\sum_{a=1}^{N-1} n_a\phi_a \tag{4.58}$$

with the bound being saturated for the two cases

$$\begin{aligned} B_i &= \mathcal{D}_i\phi \qquad \text{if } \vec{g}\cdot\vec{\phi} > 0 \\ B_i &= -\mathcal{D}_i\phi \qquad \text{if } \vec{g}\cdot\vec{\phi} < 0. \end{aligned} \tag{4.59}$$

Considering the addition of a theta term to the action, such as in the case of the $\mathcal{N} = 2$ Lagrangian, one finds that the Witten effect induces an electric charge $\vec{q} = \theta\vec{g}/2\pi$ on the monopole, which is then called a dyon.

We may also consider the BPS bound on the masses of monopoles and dyons from examining the supersymmetry algebra of $\mathcal{N} = 2$ Yang-Mills. It implies a mass bounded by the central charge Z of the algebra given by

$$M \geq \sqrt{2}|Z| = \sqrt{2}|a(n_e + \tau_{cl}n_m)| \tag{4.60}$$

where $n_e, n_m \in \mathbf{Z}$ denote the units of electric and magnetic charge and $\tau_{cl} = \frac{\theta}{2\pi} + \frac{2\pi i}{g_{YM}^2}$. Here a is the value of the gauge field A in the Higgs vacuum.

Brane construction of monopoles and dyons

Semi-classical magnetic monopole and dyons BPS states are realised in the Hanany-Witten picture by D2-branes with the topology of a disk stretching between the NS5-branes and bounded as well by two adjacent D4-branes [74,75]. More generally, magnetic monopoles may also be charged under the $U(N_f)$ flavour symmetry by means of strings extending from the D2-brane to a flavour brane.

The lift to M-theory provides us with the result that all the matter in the gauge theory is realised by open M2-branes having a boundary on the background M5-brane. For the $\mathcal{N} = 2$ case, the supergravity background has topology $\mathbf{R}^{(1,3)} \times Q^4 \times \mathbf{R}^{(3)}$, where Q^4 is a hyper-Kähler manifold spanning 4567, with x^7 the M-theory circle. If we denote by Σ the Riemann surface of the M5-brane that is holomorphically embedded in Q^4 with respect to a complex structure J, then the M2-brane shall be embedded holomorphically with respect to a distinct complex structure J'. Given a complex structure J, the set of such J' is parametrised by an S^1 which actually corresponds to the phase of the central charge of the BPS saturated state.

Monopoles are M2-branes with the topology of a disk with minimal area in their homology class. It then follows from the eleven-dimensional superalgebra that the spatial volume of the M2-brane is given by the pullback onto the brane of the holomorphic two-form Ω defined on Q^4. Provided that Ω is exact locally on the worldvolume, one can recover the Seiberg-Witten differential $\Omega = d\lambda$. The integral of this meromorphic one-form in its homology class then gives the central charge of the BPS state. In general, these M2-branes may also wrap flavour branes and acquire a charge under the flavour symmetry.

Vortices in supersymmetric gauge theory

In our results in the next chapter, we encounter an example of a BPS charge that would support a co-dimension 2 object, a vortex. Vortices are ubiquitous in physics, with many wide ranging applications. In four-dimensional theories vortices are string-like objects which carry magnetic flux through their core.

Vortices may exist in supersymmetric gauge theories, essentially through the action of the Fayet-Iliopolous term. A typical action with gauge group $U(N)$ that would support vortices is given by

$$
\begin{aligned}
S &= \int d^4x \mathrm{Tr} \frac{1}{g_{YM}^2} \left(\frac{1}{2} F^{MN} F_{MN} + (\mathcal{D}_M \phi)^2 \right) + \sum_{i=1}^{N_f} |\mathcal{D}_M q_i|^2 \\
&- \sum_{i=1}^{N_f} q_i^\dagger \phi^2 q_i - \frac{g_{YM}^2}{4} \mathrm{Tr}(\sum_{i=1}^{N_f} q_i q_i^\dagger - \nu^2 1_N)^2
\end{aligned} \tag{4.61}
$$

Spacetime indices run over $M = 0, 1, 2, 3$ and the purely spatial indices over $i = 1, 2, 3$. The action contains real scalar fields ϕ and N_f matter scalar fields q_i in the fundamental representation of $U(N)$. The D-term potential contains the Fayet-Iliopolous parameter labelled by ν^2 which induces a vev for q.

For $N_f = N$, we can view q as an $N \times N$ matrix q_i^a, where a is the colour index and i the flavour index. The unique ground state of the theory is given by $\phi = 0$ and $q_i^a = \nu \delta_i^a$. If we choose the vortex strings to lie in the x^3 direction, the scalar fields q must wind around S^1 at transverse spatial infinity. The winding number of the scalar at infinity is determined by an integer k and related to the magnetic flux B_3 in the following way:

$$
2\pi k = \mathrm{Tr} \int dx^1 dx^2 \; B_3. \tag{4.62}
$$

From solving the resulting vortex equations, it turns out that the tension of the charge k vortex is bounded by

$$
T_{|k|} \geq 2\pi^2 |k| \tag{4.63}
$$

The inequality is saturated for configurations obeying the vortex equations

$$
B_3 = \frac{g_{YM}^2}{2} (\sum_i q_i q_i^\dagger - \nu^2 1_N) \quad , \quad \mathcal{D}_z q_i = 0 \tag{4.64}
$$

where we have defined the complex co-ordinate on the transverse space $z = x^1 + ix^2$. Although no analytical solution to this equation is known, the field profiles determined numerically indicate that the energy density is localised within a core of the vortex of size $L = 1/\nu g_{YM}$ with an exponential fall off outside of this. One may also consider non-Abelian vortices and moduli spaces, as well as more complicated configurations [92].

Brane construction of vortices

Recalling our picture of Hanany-Witten models that were used to describe four-dimensional supersymmetric gauge theories, we may ask if the vortices observed in the field theory have a brane construction. The answer is yes, and they are represented by D2-branes [100]. We recall the simplest setup of two NS5-branes separated along x^6 with worldvolumes 012345 which had N D4-branes suspended between them with worldvolumes 01236. We recall we called x^7 the eleventh dimension so the transverse directions are 89(10). We can include D6-branes in between the NS5-branes with worldvolumes 012389(10).

We may now turn on the Fayet-Iliopolous term ν^2, which we choose to do along x^8. This forces the D4-branes, which we place at the origin for simplicity, to split along the D6-branes. The S-rule states that we would need N D6-branes for a zero-energy ground state since no two D4-branes are allowed to end on the same D6-brane. The Fayet-Iliopolous term is given by

$$\nu^2 = \frac{\Delta x^8}{(2\pi)^3 g_s l_s}. \tag{4.65}$$

The vortices would then be represented by D2-branes which lie along the D6-brane worldvolume that connects the two parts of the split D4-brane, that is, along 038 and with the boundary condition that it extends a finite distance and connects the two D4-brane sections. The x^3 would therefore correspond to the direction of the vortex string. We shall see in the next chapter that the central charge calculation shows that there is indeed a charge that corresponds precisely to this supersymmetric BPS state of the gauge theory, as well as the connection to structure groups.

Domain walls in supersymmetric gauge theory

There are also co-dimension 1 objects, or domain walls, in supersymmetric gauge theories [101, 102]. If we consider $\mathcal{N} = 2$ theories in four dimensions, the action that supports domain wall objects is the same as (4.61) with the addition of the mass term for the scalars given by

$$-\sum_{i=1}^{N_f} q_i^\dagger(\phi - m_i)^2 q_i \tag{4.66}$$

It turns out that for $N_f > N$ in general there exist multiple isolated vacua. This implies the existence of a domain wall which interpolates from a given vacuum Υ_- at $x^3 \to -\infty$ to a different vacuum Υ_+ at $x^3 \to +\infty$ where x^3 is chosen to be the direction transverse to the domain wall.

Upon solving the domain wall equations one finds its tension is bounded by

$$T_{\text{wall}} \geq \nu^2[\mathrm{Tr}\phi]_{-\infty}^{+\infty} \tag{4.67}$$

with saturation of the bound when

$$\mathcal{D}_3\phi = -\frac{g_{YM}^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - \nu^2) \quad , \quad \mathcal{D}_3 q_i = -(\phi - m_i)q_i \tag{4.68}$$

The situation is similar to the vortex case except we have given a vev to the N_f scalars q_i. In general, the solution to this equation is unknown but numerical methods can be employed to find qualitative features of the solution [103–105].

Brane construction of domain walls

Coming back to our Hanany-Witten model in Type IIA, and considering the case of eight conserved supercharges, domain walls must interpolate between inequivalent vacua. If we start with the same picture as before, including the D6-branes and splitting the D4-branes by the addition of a Fayet-Iliopolous term $\nu^2 \sim x^8$, then, since we also have a vev for the scalar $\phi \sim x^4$ in our action, our candidate brane must also extend in this direction.

It turns out that a D4-brane with worldvolume 01248 completely boxed in by the neighbouring NS5-branes on the ends and D6-branes and other D4-branes on the sides corresponds to a BPS domain wall in the gauge theory. Unfortunately, the worldvolume theory of these curved D4-branes is not well known and so the dynamics of domain walls are not clear in this picture. However, one can make progress by going to the strong coupling limit [106–108]

Chapter 5

Probes of wrapped M5-brane backgrounds

5.1 Central charges of an M5-brane wrapping a 2-cycle in $\mathbf{C}^2$

In this section, we proceed to construct the central charges of probe M5-branes for the first background we are examining, M5-branes wrapped on a holomorphic 2-cycle in $\mathbf{C}^2$, which is the supergravity dual of $\mathcal{N}=2$ supersymmetric gauge theories in four dimensions.

As previously mentioned, this case corresponds to a background geometry with a time-like vector K. Since our background preserves eight supercharges, the original $SU(5)$ isotropy group will be broken down, reflecting the reduced isometries of our background. These can be described by the projection conditions that our configuration satisfies. We shall discuss the structure groups that arise as a consequence of this and its implications at the end of the section.

We recall that the metric for this solution is given by

$$ds^2 = H^{-1/3}{dx^2}_{(1,3)} + 2H^{-1/3}g_{M\bar{N}}dz^M dz^{\bar{N}} + H^{2/3}{dx^2}_{(3)} \tag{5.1}$$

$$H = 4g = 4(g_{1\bar{1}}g_{2\bar{2}} - g_{1\bar{2}}g_{2\bar{1}}) \tag{5.2}$$

with conventions on the hermitian metric such that $G_{M\bar{N}} = H^{-1/3}g_{M\bar{N}} = \delta_{a\bar{b}}e^a_M e^{\bar{b}}_{\bar{N}}$.

For most of the calculation we shall work in the $z^1 = w,\ z^2 = y$ co-ordinates, taking the near-horizon limit only at the very end.

We also recall that, in the near-horizon limit, if we decompose the hermitian metric $g_{M\bar{N}}$ into tangent space zweibeins such that

$$\begin{aligned} e^1_M &= H^{1/2}\left(\partial_M F^2\right) \\ \overline{e^2_N} &= \left(\overline{\partial_N G}\right), \end{aligned} \tag{5.3}$$

then the complex structure J in which the M5-brane is embedded holomorphically is given by

$$\begin{aligned} dZ^1 &= Re\left(e^1_M dz^M\right) + iIm\left(e^1_M dz^M\right) \\ dZ^2 &= Re\left(e^2_M dz^M\right) + iIm\left(e^2_M dz^M\right). \end{aligned} \tag{5.4}$$

The projection condition this M5-brane satisfies is given by

$$\hat{\Gamma}_{0123a\bar{b}}\epsilon = i\delta_{a\bar{b}}\epsilon, \tag{5.5}$$

with $a, b = 1, 2$. It is also well known that this background admits a supersymmetric M2-brane probe which is a BPS state of the worldvolume theory of the M5-brane. Since the complex submanifold in which we are embedding this brane is actually hyper-Kähler (as are all two complex-dimensional Kähler manifolds), this geometry admits a family of inequivalent complex structures parametrised by a two-sphere S^2, with $SU(2)$ commutation relations between them. Also, in four dimensions, the hyper-Kähler condition implies Ricci flatness and should therefore admit a covariantly constant holomorphic two-form.

In order to ensure that the two-brane ends on the five-brane, we shall need to wrap the M2-brane on a holomorphic cycle with respect to a complex structure J' which is orthogonal to the complex structure J in which the background M5-brane was embedded holomorphically. Given a complex structure J, the set of such J' for a hyper-Kähler manifold is parametrised by an S^1 that actually corresponds to the phase of the central charge of the BPS saturated state [74].

In terms of the M5-brane holomorphic coordinates, the projection condition for the M2-brane can be written

$$\mathcal{P}\epsilon = \left(e^{i\phi}\hat{\Gamma}_{0ab} + e^{-i\phi}\hat{\Gamma}_{0\bar{a}\bar{b}}\right)\epsilon = \epsilon. \tag{5.6}$$

We have included the arbitrary phase ϕ for generality. We note that the linear combination of holomorphic and anti-holomorphic projection conditions do indeed insure it is an Hermitian projector with $\mathcal{P}^2 = 1$. This additional constraint cuts the number of supersymmetries by half (leaving four real supersymmetries), confirming that the M2-brane is a BPS object of the M5-brane worldvolume gauge theory.

The next thing to notice is that using the identity $\hat{\Gamma}_{0123456789(10)} \equiv 1$ we can show that our projections actually allow for another M5-brane that does not break any further supersymmetries. We find it wraps a holomorphic cycle with respect to a complex structure J'' which is orthogonal to both the previous cases and exhausts the three independent complex structures of the hyper-Kähler manifold they are embedded in. In addition, this M5-brane has a worldvolume extension along the 89(10) space. Explicitly, the projection condition for this "hidden" M5-brane is

$$\left(e^{i(\phi+\pi/2)}\hat{\Gamma}_{0ab} + e^{-i(\phi+\pi/2)}\hat{\Gamma}_{0\bar{a}\bar{b}}\right)\hat{\Gamma}_{89(10)}\epsilon = \epsilon \tag{5.7}$$

Finally, there is another projection condition which is compatible and commutes with these three (and therefore does not break supersymmetry any further). This can best be expressed by defining the complex coordinates

$$\begin{aligned}
\xi^1 &= e^1 + ie^{(10)} \\
\xi^2 &= e^2 + ie^9 \\
\xi^3 &= e^3 + ie^8
\end{aligned} \tag{5.8}$$

which allows us to write the projection condition in the following way:

$$\hat{\Gamma}_{0\xi^i\xi^{\bar{j}}}\epsilon = i\delta_{i\bar{j}}\epsilon, \tag{5.9}$$

with $i, j = \xi^1, \xi^2, \xi^3$. As before we have $\delta_{1\bar{1}} = 1/2$. That this additional complex submanifold is compatible with all our earlier projections somehow reflects that the original, larger $SU(5)$ isometry group has been broken down by the appearance of branes in the geometry. This is consistent with the known structure groups of our background, as we shall discuss. We can show this more clearly by combining the preceding constraints to build the following projection, which shows some of that residual structure:

$$\left(\hat{\Gamma}_{0\xi^1\xi^2\xi^3 a\bar{b}} + \hat{\Gamma}_{0\bar{\xi}^1\bar{\xi}^2\bar{\xi}^3 a\bar{b}}\right)\epsilon = \frac{i}{4}\delta_{a\bar{b}}\epsilon. \tag{5.10}$$

One can check that this projection condition includes the original background M5-brane projection (5.5). It is also reminiscent of an M5-brane wrapping a special Lagrangian 3-cycle in one manifold $\tilde{M}_6$, and a holomorphic 2-cycle in the hyper-Kähler manifold M_4. We shall expand on this geometrical statement a little later, where we shall see that $\tilde{M}_6$ will turn out to have a reduced group structure. We discuss possible field theory interpretations at the end of the section.

These conditions then complete the set of independent, commuting projections and thus determine a unique spinor up to scale. The scale of the spinor, which we will use shortly to calculate the forms K, Ω and Σ, is given by fixing $\epsilon^\dagger\epsilon = \Delta$. Using the fact that K is a Killing vector of our background, we found that $\Delta = \sqrt{-g_{00}}$. We can now proceed to calculate the non-trivial components of each form. A quick calculation reveals that $K_i = 0$ for $\{i = 1, 2, 3, a, \bar{b}, 8, 9, (10)\}$, since, for example,

$$\begin{aligned} \hat{K}_1 &= \bar{\epsilon}\hat{\Gamma}_1\epsilon = \bar{\epsilon}\hat{\Gamma}_1\left(e^{i\phi}\hat{\Gamma}_{0ab} + e^{-i\phi}\hat{\Gamma}_{0\bar{a}\bar{b}}\right)\epsilon = -\bar{\epsilon}\left(e^{i\phi}\hat{\Gamma}_{01ab} + e^{-i\phi}\hat{\Gamma}_{01\bar{a}\bar{b}}\right)\epsilon \\ &= 0 \end{aligned}$$

where in the second step we have used the M2-brane projection condition and in the last step the fact that four-forms constructed in this way vanish identically.

After some work, the resulting forms turn out to be:

$$\begin{aligned}
K &= -H^{-1/3}dt && (5.11)\\
\Omega &= 1/2\left(e^{-i\phi}dz^1\wedge dz^2+e^{i\phi}d\bar{z}^1\wedge d\bar{z}^2\right)+iH^{-1/6}\delta_{i\bar{j}}\,\xi^i\wedge\xi^{\bar{j}} && (5.12)\\
\Sigma &= H/2\left(e^{-i(\phi+\pi/2)}dz^1\wedge dz^2+e^{i(\phi+\pi/2)}d\bar{z}^1\wedge d\bar{z}^2\right)\wedge dx^8\wedge dx^9\wedge dx^{(10)}\\
&\quad +iH^{-1/2}g_{M\bar{N}}\,Re\left(\xi^1\wedge\xi^2\wedge\xi^3\right)\wedge dz^M\wedge dz^{\bar{N}}\\
&\quad -1/4\,dt\wedge dz^1\wedge d\bar{z}^1\wedge dz^2\wedge d\bar{z}^2\\
&\quad -H^{-1/3}\delta_{i\bar{j}}\delta_{k\bar{l}}\,dt\wedge\xi^i\wedge\xi^{\bar{j}}\wedge\xi^k\wedge\xi^{\bar{l}}. && (5.13)
\end{aligned}$$

The last two terms above do not play a role for possible static probe branes but we can consider taking the Hodge dual. These could be part of the D6-brane central charge in the Type IIA ten-dimensional picture, which is only geometry in eleven dimensions.

Now we recall that in order to calculate the central charge we also need the six and three-form potentials for our particular background. The six-form potential can be ascertained easily if we remember that the background M5-brane satisfies a generalised calibration (2.37) which, by virtue of the BPS supersymmetry condition, is gauge equivalent to the spacetime gauge potential under which it is charged. We proceed by working with the asymptotically flat background co-ordinates z^i before taking the near-horizon limit. Taking into account that the potential vanishes at spatial infinity, and taking the contraction with respect to K, we can conclude that

$$\imath_K C = -i(H^{-1}g_{M\bar{N}}-\delta_{M\bar{N}})\,dx^1\wedge dx^2\wedge dx^3\wedge dz^M\wedge dz^{\bar{N}} \tag{5.14}$$

The only thing that remains is to define the three-form A since, as in this case it is a magnetic potential, it is not globally well defined. The natural solution is to define the integral of $A\wedge\Omega$ over the spatial worldvolume of the brane such that

$$\int_{M5} F\wedge\Omega=\int_{\partial M5} A\wedge\Omega. \tag{5.15}$$

Direct calculation reveals that the product is given by

$$
\begin{aligned}
\Omega \wedge F &= -1/2\, dH \wedge \left(e^{-i(\phi+\pi/2)} dz^1 \wedge dz^2 + e^{i(\phi+\pi/2)} d\bar{z}^1 \wedge d\bar{z}^2\right) \wedge \\
&\quad dx^8 \wedge dx^9 \wedge dx^{(10)} \\
&\quad +(dx^1 \wedge dx^{(10)} + dx^2 \wedge dx^9 + dx^3 \wedge dx^8) \wedge \\
&\quad i\epsilon_{\alpha\beta\gamma}\partial_\gamma g_{M\bar{N}} dz^M \wedge dz^{\bar{N}} \wedge dx^\alpha \wedge dx^\beta
\end{aligned} \tag{5.16}
$$

so we can deduce that the magnetic potential A can be defined,

$$
\begin{aligned}
\Omega \wedge A &= -1/2\,(H-1)\left(e^{-i(\phi+\pi/2)} dz^1 \wedge dz^2 + e^{i(\phi+\pi/2)} d\bar{z}^1 \wedge d\bar{z}^2\right) \wedge \\
&\quad dx^8 \wedge dx^9 \wedge dx^{(10)} \\
&\quad -i\left(dx^1 \wedge dx^8 \wedge dx^9 + dx^3 \wedge dx^9 \wedge dx^{(10)}\right. \\
&\quad \left. -dx^2 \wedge dx^8 \wedge dx^{(10)}\right) \wedge (g_{M\bar{N}} - \delta_{M\bar{N}}) dz^M \wedge dz^{\bar{N}}.
\end{aligned} \tag{5.17}
$$

We can see straight away from this expression that the contraction $\imath_K A$ vanishes. We note that we can define the (1,1)-form $J = \frac{i}{2}(e^1 \wedge e^{\bar{1}} + e^2 \wedge e^{\bar{2}})$ on the M_4 manifold and also define a (1,1)-form on $\tilde{M}_6$ by $\tilde{J} = \frac{i}{2}(\xi^1 \wedge \xi^{\bar{1}} + \xi^2 \wedge \xi^{\bar{2}} + \xi^3 \wedge \xi^{\bar{3}})$. The justification for this will be discussed as the end of the section in terms of structure groups. We may also define the flat space Kähler form $J_f = i\delta_{a\bar{b}} dz^a \wedge dz^{\bar{b}}$ on M_4 and also the flat holomorphic three-form $\tilde{\Psi}_{3f} = (dx^1 + idx^{(10)}) \wedge (dx^2 + idx^9) \wedge (dx^3 + idx^8)$ and the flat Kähler form $\tilde{J}_f = iH^{-1/6}\delta_{i\bar{j}}\xi^i \wedge \xi^{\bar{j}} = H^{-1/6}J$ on $\tilde{M}_6$.

Taking this into account and assembling all the terms, changing w, y to F^2, G where appropriate, we get that the central charges on an M5-brane probe of an M5-brane wrapping a holomorphic 2-cycle in $\mathbf{C}^2$ is given by

$$
\begin{aligned}
\int K^M P_M &\geq \mp \int (\imath_K C + \Sigma + (A + dB) \wedge \Omega) \\
&\geq \mp \int \Big(Re(\tilde{\Psi}_{3f}) \wedge J_f \\
&\quad + \frac{1}{2}\left(e^{-i(\phi+\pi/2)} dF^2 \wedge dG + e^{i(\phi+\pi/2)} d\bar{F}^2 \wedge d\bar{G}\right) \wedge dx^8 \wedge dx^9 \wedge dx^{(10)} \\
&\quad + dt \wedge\ J_f \wedge J_f \\
&\quad + dt \wedge\ \tilde{J}_f \wedge \tilde{J}_f \\
&\quad + dB \wedge \Omega \Big).
\end{aligned} \tag{5.18}
$$

We note that the supersymmetry algebra is unaltered from flat space for a suitable choice of co-ordinates, of the same form as (4.13). The first term indicates the obvious possibility that an M5-brane probe which is parallel to the background M5-brane is an allowed supersymmetric probe. There are also other possibilities such as an M5-brane with spatial embedding 2458(10) for example. Depending on the boundary conditions, these may have field theory interpretations.

In addition, the second term allows for an M5-brane which is embedded holomorphically in the hyper-Kähler manifold with respect to J'' and extended along 89(10).

The possible holomorphy conditions on the pullback onto the probe branes are given by

$$\begin{aligned} H^{1/2}\frac{\partial F^2}{\partial\sigma^1} &= e^{i\phi}\frac{\partial\bar{G}}{\partial\sigma^2} \\ H^{1/2}\frac{\partial F^2}{\partial\sigma^2} &= -e^{i\phi}\frac{\partial\bar{G}}{\partial\sigma^1}, \end{aligned} \tag{5.19}$$

where we have defined $\sigma = \sigma^1 + i\sigma^2$ to be the complex coordinate on the probe worldvolume . We have chosen the different complex structures J, J' and J'' so they are specified by $\phi = 0, \pi/2, -\pi/2$, respectively.

This probe would be related to the M2-brane probe which gives the masses of BPS states in the worldvolume gauge theory of the background M5-brane under appropriate boundary conditions. In our present notation, we recall that the central charge of the probe M2-brane would be given by

$$\mp\int\left(1/2\left(e^{-i\phi}dF^2\wedge dG + e^{i\phi}d\bar{F}^2\wedge d\bar{G}\right) + iH^{-1/6}\tilde{J}\right). \tag{5.20}$$

If we look at the first term above, the difference between the two would be a volume modulus of the 89(10) space and also a rotation in the complex structure. Depending on the boundary conditions, this extra volume modulus could well be finite, like in the vortex case.

The second term above $iH^{-1/6}\tilde{J}$ is a calibration form of the $\tilde{M}_6$ manifold and denotes the possibility of co-dimension two objects on the worldvolume theory. These turn out to correspond to BPS vortices, something that can pictured in terms of

Hanany-Witten models [92]. As mentioned in the last chapter, the boundary condition along the totally transverse directions 89(10) would mean that these vortices would have finite tension.

The last couple of terms of the M5-brane central charge cannot be pulled back consistently to a static probe brane so we may consider taking the Hodge dual of them. We note, however, that the quantities in the brackets $(H^{-1/3}\ J \wedge J)$ and $(H^{-1/3}\ \tilde{J} \wedge \tilde{J})$ are calibrating forms. Taking the dual then gives terms that would contribute to the Type IIA ten-dimensional central charges for the D6-brane, which in M-theory is given by pure geometry. These terms can be re-written

$$\begin{aligned}
\Sigma_{*(0JJ)} &= -*\left(H^{-1/6}\, e^0 \wedge J \wedge J\right) \\
&\quad -*\left(H^{-1/6}\, e^0 \wedge \tilde{J} \wedge \tilde{J}\right) \\
&= -H^{-1/6}\, \tilde{J} \wedge \tilde{J} \wedge \tilde{J} \\
&\quad -H^{-1/6}\, \tilde{J} \wedge J \wedge J.
\end{aligned} \tag{5.21}$$

We see that there are only terms which give the volumes of $\tilde{M}_6$ and M_4 respectively. Since we are compactifying along the M-theory circle (contained in M_4), only the first term would contribute to the D6-brane central charge.

To make sense of the various forms we have defined and arguments about the manifold $\tilde{M}_6$ we have to talk about structure groups. To our knowledge, there is no complete classification for eleven dimensional backgrounds with a time-like Killing spinor and flux, of which the brane configuration studied in this section is an example. However, from what is known from holonomy groups of M-theory vacua with no flux [90], we can deduce the structure group for our configuration. It is easy to see that, for the case of our M5-brane wrapped on a 2-cycle in $\mathbf{C}^2$ with the BPS M2-brane probe ending on it and wrapped on a 2-cycle in a different complex structure, we preserve $\frac{1}{8}$ of supersymmetry. Backgrounds preserving this fraction of supersymmetry allow for three possible structure groups, but we can deduce that the appropriate one for our case is that we should have an overall $SU(2) \times SU(3)$ structure. This fits in with the known M_4 manifold typical of these spacetimes, whilst uncovering the $\tilde{M}_6$ manifold which was hinted at by the allowed projection conditions. The fact that M_6 has an $SU(3)$ structure then means that, although it is no longer Calabi-Yau (since this requires it to have $SU(3)$ holonomy), it is still a complex manifold. We can therefore use a similar technique to [64], to recover calibration forms for this manifold and for the product manifold $M_4 \times \tilde{M}_6$.

As we shall see in the next section, where the classification of eleven dimensional backgrounds with flux and a null Killing spinor has been done, these structure groups provide en elegant illustration of the transitions of the wrapped M5-brane's worldvolume, which give rise to intersecting BPS domain wall configurations on its worldvolume theory.

Another way to look at our background is to say that it is globally of the form $\mathbf{R} \times \mathbf{C}^2 \times \tilde{\mathbf{M}}_6$, since there seems to be an allowed almost complex structure definable on $\tilde{M}_6$. Looking at it this way, the background M5-brane wraps an associative 3-cycle in $\tilde{M}_6$, in particular $Re(\Psi_{(3)})$ in our conventions, where $\Psi_{(3)}$ is the (3,0)-form $\Psi_{(3)} = \xi^1 \wedge \xi^2 \wedge \xi^3$. This is consistent with the known fact that an M5-brane wrapping an associative 3-cycle in a Calabi-Yau 3-fold spanning 12389(10) has a calibrating form $\Upsilon = iH^{-1/6}\tilde{J}$.

Lastly, we would like to note that, in the first instance, one can probe the background with an M5-brane embedded holomorphically with respect to J' with the projection condition

$$\left(e^{i\phi}\hat{\Gamma}_{0ab} + e^{-i\phi}\hat{\Gamma}_{0\bar{a}\bar{b}}\right)\hat{\Gamma}_{89(10)}\epsilon = \epsilon. \tag{5.22}$$

Then we would find that, given the identity $\hat{\Gamma}_{0123456789(10)} \equiv 1$, there was a "hidden" M2-brane embedded holomorphically with respect to $-J''$ with the projection condition

$$\left(e^{i(\phi-\pi/2)}\hat{\Gamma}_{0ab} + e^{-i(\phi-\pi/2)}\hat{\Gamma}_{0\bar{a}\bar{b}}\right)\epsilon = \epsilon. \tag{5.23}$$

The corresponding central charge for this brane would be

$$\mp \int \left(e^{-i(\phi-\pi/2)}dF^2 \wedge dG + e^{i(\phi-\pi/2)}d\bar{F}^2 \wedge d\bar{G} + \ldots\right). \tag{5.24}$$

and the relevant term in the M5-brane central charge would become

$$\mp \int (\ldots + (e^{-i\phi} dF^2 \wedge dG + e^{i\phi} d\bar{F}^2 \wedge d\bar{G}) \wedge dx^8 \wedge dx^9 \wedge dx^{(10)} \ldots) \tag{5.25}$$

which are the same results as before except that the rotation of complex structures is in the opposite direction to what we had previously. This would be interpreted as an anti-M2-brane for instance. It makes no qualitative difference to the answer though.

To help visualise the branes at some limit we can make a table. The singular limit of these wrapped M5-branes on holomorphic 2-cycles is given by orthogonally intersecting five-branes, which we lay out here to make the setup more transparent. The worldvolume directions spanned by the M5-branes that source the background are indicated by $\otimes$, with the allowed probe branes having worldvolume directions denoted by $\odot$. We have drawn double vertical lines to point out the $\tilde{M}_6$ manifold spanned by 12389(10). We have also drawn single vertical lines to denote the $\mathbf{C}^2$ subspace, which contains probes wrapped on all three complex structures, as can be seen from the middle entries. This singular limit shows the probe M2-brane wrapped on J', and the "hidden" M5-brane wrapped on J''. Also shown are the M2-brane corresponding to BPS vortices (which wrap a holomorphic 2-cycle in $\tilde{M}_6$) and $*$ D6 denotes the object that would correspond to a D6-brane in Type IIA string theory.

	0	1	2	3	$\Re(G)$	$\Im(G)$	$\Re(F^2)$	$\Im(F^2)$	8	9	10
M5	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$					
M5	$\otimes$	$\otimes$	$\otimes$	$\otimes$			$\otimes$	$\otimes$			
M2	$\odot$				$\odot$		$\odot$				
M2	$\odot$	$\odot$									$\odot$
M5	$\odot$		$\odot$		$\odot$	$\odot$			$\odot$		$\odot$
M5	$\odot$				$\odot$			$\odot$	$\odot$	$\odot$	$\odot$
$*$ D6	$\odot$	$\odot$	$\odot$	$\odot$					$\odot$	$\odot$	$\odot$

Table 5.1: Some possible supersymmetric probe embeddings. We have denoted the real and imaginary parts of our coordinates such that $G = \Re(G) + i\Im(G)$.

5.2 Central charges of an M5-brane wrapping a 2-cycle in $\mathbf{C}^3$

In this section we calculate the central charge for the last background we are examining, M5-branes wrapped on a holomorphic 2-cycle in $\mathbf{C}^3$, which is the supergravity dual of $\mathcal{N} = 1$ MQCD supersymmetric gauge theories in four dimensions.

As opposed to the previous example, this case corresponds to a background geometry with a null vector K. Since our background preserves four supercharges, the original $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ isotropy group will be broken down, reducing the isometries of our background. Thus, our projection conditions will differ in structure from those in the last section, reflecting the different isometries of this geometry.

We recall that the form of the metric for this solution is

$$\begin{aligned} ds^2 &= H^{-1/3}dx^2{}_{(1,3)} + 2H^{1/6}g_{M\bar{N}}dz^M dz^{\bar{N}} + H^{2/3}dy^2 \\ \det g &= H, \end{aligned} \tag{5.26}$$

and the metric conventions $G_{M\bar{N}} = H^{1/6}g_{M\bar{N}} = \delta_{a\bar{b}}e^a_M e^{\bar{b}}_{\bar{N}}$ for $a, b = 1, 2, 3$ and $\delta_{1\bar{1}} = 1/2$. We shall write the vielbein e^{z^1} to avoid confusion with the x^1 part of the metric.

To start with, the projection condition satisfied by the background M5-brane wrapped on a holomorphic 2-cycle is given by

$$\hat{\Gamma}_{0123a\bar{b}}\epsilon = i\delta_{a\bar{b}}\epsilon, \tag{5.27}$$

where now we recall that $a, b = 1, 2, 3$ (since $\mathbf{C}^3$ is defined in the $\{456789\}$ space). We also find that we can choose a compatible projection choice along the 01 directions of the form

$$\hat{\Gamma}_{01}\epsilon = \pm\epsilon \tag{5.28}$$

where the ambiguity of sign comes from the requirement that the projector squares to 1. This is also equivalent to adding momentum along the 1 direction, and is well known to break a further $\frac{1}{2}$ supersymmetry. Similarly, for the 23 and z_1, z_2, z_3

spaces, we have a certain freedom to impose compatible projections. If we permit an arbitrary angle in the 23 plane and an arbitrary phase for the z_1, z_2, z_3 space, we have

$$\left(e^{i\phi}\left(\alpha\hat{\Gamma}_2 - \beta\hat{\Gamma}_3\right)\hat{\Gamma}_{z_1 z_2 z_3} + e^{-i\phi}\left(\alpha\hat{\Gamma}_2 - \beta\hat{\Gamma}_3\right)\hat{\Gamma}_{\overline{z_1 z_2 z_3}}\right)\epsilon = \epsilon \qquad (5.29)$$

with the condition that $\alpha^2+\beta^2 = 1$ and we can check this projector is also Hermitian, as is required. Again, the equations for 1/8-SUSY hold for arbitrary phase ϕ.

Finally, one should note that using the identity $\hat{\Gamma}_{0123456789y} \equiv 1$ we can show that our projections imply:

$$\hat{\Gamma}_y \epsilon = -\epsilon. \qquad (5.30)$$

These provide a set of independent, commuting projections which determine a unique spinor up to scale. The scale of the spinor in this case is fixed, again using the fact that K is a Killing vector, to be

$$\epsilon^\dagger \epsilon = \Delta = H^{-1/6}. \qquad (5.31)$$

As in the previous section, we can now proceed to calculate the non-trivial components of each form. For example, in this case we can easily show that the $K_i(i = 2, 3), K_a, K_{\bar{b}}(a, b = 1, 2, 3)$ and K_y components vanish since, for example,

$$\hat{K}_2 \;=\; \bar{\epsilon}\hat{\Gamma}_2\epsilon \;=\; \pm\bar{\epsilon}\hat{\Gamma}_2\hat{\Gamma}_{01}\epsilon \;=\; \pm\bar{\epsilon}\hat{\Gamma}_{012}\epsilon \;=\; 0$$

where in the second step we have used the $\hat{\Gamma}_{01}$ projection condition (5.28), and in the last step the fact that the three-form vanishes identically.

Computing the rest of the forms in a similar manner, we find the following:

$$K = -H^{-1/3}\left(dt \mp dx^1\right) \tag{5.32}$$

$$\Omega = \left(dt \mp dx^1\right) \wedge dy \tag{5.33}$$

$$\begin{aligned}\Sigma = & \mp i H^{-1/2} g_{M\bar{N}} \left(dt \mp dx^1\right) \wedge dx^2 \wedge dx^3 \wedge dz^M \wedge dz^{\bar{N}} \\ & + \left(dt \mp dx^1\right) \wedge (\omega \wedge \omega) \\ & - \frac{H^{-1/2}}{2} e^{-i\phi} (\alpha \mp i\beta) \left(dt \mp dx^1\right) \wedge dx^2 \wedge \Psi_{(3)} \\ & - \frac{H^{-1/2}}{2} e^{i\phi} (\alpha \pm i\beta) \left(dt \mp dx^1\right) \wedge dx^2 \wedge \bar{\Psi}_{(3)} \\ & \pm \frac{iH^{-1/2}}{2} e^{-i\phi} (\alpha \mp i\beta) \left(dt \mp dx^1\right) \wedge dx^3 \wedge \Psi_{(3)} \\ & \mp \frac{iH^{-1/2}}{2} e^{i\phi} (\alpha \pm i\beta) \left(dt \mp dx^1\right) \wedge dx^3 \wedge \bar{\Psi}_{(3)}.\end{aligned} \tag{5.34}$$

We have denoted the holomorphic three-form as $\Psi_{(3)} = e^{z^1} \wedge e^{z^2} \wedge e^{z^3}$. The last four terms $\Sigma_{K(2+3)(\Psi+\bar{\Psi})}$ can be simplified further by noting that we can let $\alpha = \cos\theta$ and $\beta = \sin\theta$, which results in

$$\begin{aligned}\Sigma_{K(2+3)(\Psi+\bar{\Psi})} = & -\frac{H^{-1/2}}{2} \left(dt \mp dx^1\right) \wedge e^{\mp i\theta} \left(dx^2 \mp i dx^3\right) \wedge e^{-i\phi} \Psi_{(3)} \\ & -\frac{H^{-1/2}}{2} \left(dt \mp dx^1\right) \wedge e^{\pm i\theta} \left(dx^2 \pm i dx^3\right) \wedge e^{i\phi} \bar{\Psi}_{(3)}.\end{aligned} \tag{5.35}$$

Furthermore, if we choose for convenience the bottom signs in the lines above and define the complex co-ordinate $\lambda = x^2 + ix^3$, then we can re-write it in the following way:

$$\begin{aligned}\Sigma_{K(\lambda\Psi+\bar{\lambda}\bar{\Psi})} = & -\frac{H^{-1/2}}{2} \left(dt \mp dx^1\right) \wedge e^{i(\theta-\phi)} \left[d\lambda \wedge \Psi_{(3)}\right] \\ & -\frac{H^{-1/2}}{2} \left(dt \mp dx^1\right) \wedge e^{-i(\theta-\phi)} \left[d\bar{\lambda} \wedge \bar{\Psi}_{(3)}\right].\end{aligned} \tag{5.36}$$

In order to find the constraint on θ and ϕ, it is useful to express this fully in terms of vielbeins and substitute in for K. We find

$$\Sigma_{K(\Psi_{(4)}+\bar{\Psi}_{(4)})} = 1/2 \left[K \wedge e^{i(\theta-\phi)} \Psi_{(4)} + K \wedge e^{-i(\theta-\phi)} \bar{\Psi}_{(4)}\right]. \tag{5.37}$$

We have defined the holomorphic four-form $\Psi_{(4)} = e^\lambda \wedge \Psi_{(3)}$ in the enlarged $\mathbf{R}^7 \times S^1$ subspace of the M-theory vacuum. Since this five-form Σ should be real, and that it is determined uniquely [77] given our projections conditions, we find that

$$\Sigma_{K\wedge Re(\Psi_{(4)})} = +K \wedge Re\left(e^{i(\theta-\phi)}\Psi_{(4)}\right), \tag{5.38}$$

which is of the same form as in [77] and $Re\left(\Psi_{(4)}\right) \subset \Phi_{(4)}$, with $\Phi_{(4)}$ the Cayley four-form. This implies that for any two supersymmetric brane probes of this type, the relative angle θ between them in the $\lambda\bar{\lambda}$-plane should be equal to the angle ϕ in the 89-plane.

Proceeding in the same manner as before, we now need to calculate the six and three-form potential which are necessary to find the central charge. Once more the six-form is easy to recover since it is gauge equivalent to the calibration bound satisfied by the background M5-brane (2.42). If we make sure to include the right asymptotic conditions and contract with K we find that

$$\imath_K C = \pm i(H^{-1/2}g_{M\bar{N}} - \delta_{M\bar{N}})\left(dt \mp dx^1\right) \wedge dx^2 \wedge dx^3 \wedge dz^M \wedge dz^{\bar{N}}. \tag{5.39}$$

We can obtain the three-form magnetic potential in the same manner as before, defining the appropriate integral. We find that

$$\begin{aligned} \Omega \wedge F &= \left(dt \mp dx^1\right) \wedge dy \wedge \partial_y(\omega \wedge \omega) \\ &= \left(dt \mp dx^1\right) \wedge d(\omega \wedge \omega) \end{aligned} \tag{5.40}$$

which gives us an expression for the three-form potential of the form

$$\begin{aligned} \Omega \wedge A &= -\left(dt \mp dx^1\right) \wedge (\omega \wedge \omega) \\ &= +\left(dt \mp dx^1\right) \wedge (g_{M[\bar{N}}g_{|P|\bar{Q}]} - \delta_{M\bar{N}}\delta_{P\bar{Q}})dz^M \wedge dz^{\bar{N}} \wedge dz^P \wedge dz^{\bar{Q}} \end{aligned} \tag{5.41}$$

When combining this term with the second term from the expression for Σ, we find, after some cancellations, that we are left with something of the form $-\delta_{M\bar{N}}\delta_{P\bar{Q}}(dt \mp dx^1) \wedge dz^M \wedge dz^{\bar{N}} \wedge dz^P \wedge dz^{\bar{Q}}$. Defining the (1,1)-form on the Hermitian manifold

$M_6 \subset \mathbf{C}^3$ by $J_f = i\delta_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ and noting that the flat holomorphic three-form is given by $\Psi_{(3)f} = H^{-1/2}\Psi_{(3)}$, as before, we can simplify this expression somewhat. We also note that as before the contraction $\imath_K A$ also vanishes for this background. We can now compile all the terms that make up the central charge, which yields

$$\begin{aligned}
\int K^M P_M \ &\geq \ \mp \int (\imath_K C + \Sigma + (A + dB) \wedge \Omega) \\
&\geq \ \mp \int \Big(\mp i \left(dt \mp dx^1\right) \wedge dx^2 \wedge dx^3 \wedge J_f \\
&\quad + (dt \mp dx^1) \wedge J_f \wedge J_f \\
&\quad - \frac{1}{2}\left(dt \mp dx^1\right) \wedge e^{i\theta} d\lambda \wedge e^{-i\phi} H^{-1/2} \Psi_{(3)} \\
&\quad - \frac{1}{2}\left(dt \mp dx^1\right) \wedge e^{-i\theta} d\bar{\lambda} \wedge e^{i\phi} H^{-1/2} \bar{\Psi}_{(3)} \\
&\quad + dB \wedge \Omega \Big). \qquad (5.42)
\end{aligned}$$

So again we see that all the terms actually combine to give us the flat-space supersymmetry algebra, in the form (4.17). Namely, the expression for the central charges takes the form

$$\left(dt \mp dx^1\right) \wedge \Phi_{(4f)} + dB \wedge \Omega \qquad (5.43)$$

where $\Phi_{(4f)}$ is the flat-space Cayley four-form.

The first possibility allowed by the central charge, for a suitable embedding, is the obvious case of a parallel probe brane. The next term allows for a probe wrapped on a holomorphic 4-cycle in $\mathbf{C}^3$. Of more interest are the next two terms. We have written them in a suggestive manner which we will explain shortly. These central charges are equivalent to a probe M5-brane wrapping a Cayley calibrated 4-cycle in some manifold M_8. This has a natural interpretation as intersecting MQCD domain walls preserving 1/16 of the overall supersymmetry. They are thus 1/2 BPS states of the worldvolume theory. We can further add some momentum along the 01 directions (or along the null Killing vector) which again breaks half the supersymmetries leaving us with 1/32. The domain wall interpretation can be illustrated by the sequence

$$\mathbf{R}^{(1,1)} \times Re\left(\Psi_{(4)}\right) \rightarrow \mathbf{R}^{(1,2)} \times \Psi_{(3)} \rightarrow \mathbf{R}^{(1,3)} \times \Sigma_2 \tag{5.44}$$

which was made explicit in our construction. The 4-cycle calibrated by $Re\left(\Psi_{(4)}\right)$ contains a line in the 23 $(\lambda\bar{\lambda})$ plane and also a 3-cycle calibrated by $\Psi_{(3)}$, where $\Psi_{(3)}$ is an associative 3-cycle. These are the individual domain walls. As one moves away from them, the space should change to $\mathbf{R}^{(1,3)} \times \Sigma_2$, representing the different vacua of the theory.

We can now justify this argument as follows. From the classification of eleven dimensional supergravity with a background null Killing spinor and flux [45], we can see that this is indeed the case. We started out with a background of an M5-brane wrapped on a 2-cycle in $\mathbf{C}^3$. We can see that since this preserves 1/8 supersymmetry, this background has a $(SU(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$ structure, as we would expect. This confirms, as before, that we are dealing with a complex manifold on which we can define a holomorphic three-form. In the singular limit, these three M5-branes will represent the vacua of our domain wall configuration.

We then saw, from the central charge result, that this background allows for a 1/2 BPS M5-brane probe with worldvolume $\mathbf{R}^{(1,1)} \times Re\left(\Psi_{(4)}\right)$, which was interpreted as a BPS intersecting domain wall on the worldvolume theory. This configuration now preserves 1/16 supersymmetry. We can see that we were coherent in claiming that this M5-brane wraps a Cayley calibrated 4-cycle since we now have a $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure according to the classification. Finally, adding momentum along the Killing direction 01 breaks a further 1/2 supersymmetry, and accordingly, since we only have one supersymmetry generator left in our background, we have recovered the expected $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure. So the geometrical description fits in rather nicely with the field theory interpretation.

Furthermore, our constraint on the θ and ϕ phase spaces has a simple interpretation. The vector representing the intersection angle of the domain walls in the 23 space should be of the same magnitude as the vector representing the angle between them in the 'electric/magnetic' charge space. In our construction this was the 89 space. This implies that, for example, we would have $2468 \rightarrow -3469$, which agrees with the standard form of the Cayley four-form. This agrees, up to conventions, with the results of [109].

Lastly, we can calculate the tension of these domain walls. From the way we wrote the term, it is natural to conclude that it is given by

$$T_{DM} = \left| e^{-i\phi} \int H^{-1/2} \Psi_{(3)} \right| . \qquad (5.45)$$

This is consistent with the recent result [64] where it was found that $H^{-1/2}\Psi_{(3)}$ corresponds to a calibrating form in the geometry, and is thus closed. Therefore the integral represents a topological charge and we can conclude that these domain walls we have constructed are stable and have finite tension.

The singular limit of these wrapped M5-branes on holomorphic 2-cycles is given by orthogonally intersecting five-branes, which we lay out here to make the setup more transparent. The worldvolume directions spanned by the M5-branes that source the background are indicated by $\otimes$, with the allowed probe branes having worldvolume directions denoted by $\odot$. We have drawn double lines to separate the $\mathbf{R}^7 \times S^1$ subspace from the rest. The fourth entry denotes a probe wrapped on the square of the Kähler form. The bottom two entries clearly show an example of a Cayley calibrated 4-cycle. Within this space, delimited by a single vertical line, is the $\mathbf{C}^3$ subspace, which contains the associative 3-cycle as can also be seen from the bottom two entries. This singular limit shows two domain walls intersecting along the 01 direction (with momentum running along this direction) and making a $\pi/2$ angle to each other in the 23 plane and $-\pi/2$ angle in the 89 plane.

	0	1	2	3	$\Re(G)$	$\Im(G)$	$\Re(F^2)$	$\Im(F^2)$	$\Re(E^2)$	$\Im(E^2)$	10
M5	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$					
M5	$\otimes$	$\otimes$	$\otimes$	$\otimes$			$\otimes$	$\otimes$			
M5	$\otimes$	$\otimes$	$\otimes$	$\otimes$					$\otimes$	$\otimes$	
M5	$\odot$	$\odot$			$\odot$	$\odot$			$\odot$	$\odot$	
M5	$\odot$	$\odot$	$\odot$		$\odot$		$\odot$		$\odot$		
M5	$\odot$	$\odot$		$\odot$	$\odot$		$\odot$			$\odot$	

Table 5.2: Some possible supersymmetric probe embeddings. We have denoted the real and imaginary parts of our coordinates such that $G = \Re(G) + i\Im(G)$.

5.3 Discussion

In this chapter we have calculated the central charges on the worldvolume of M2- and M5-branes probes in a background of M5-branes which are wrapped on 2-cycles in $\mathbf{C}^2$ and $\mathbf{C}^3$. This has revealed what supersymmetric M-brane probes of the eleven-dimensional supergravity solutions [31, 46, 48] are allowed. These probes have revealed interesting features about the corresponding $\mathcal{N} = 2$ and $\mathcal{N} = 1$ field theories.

In the case of the background sourced by M5-branes wrapping a holomorphic 2-cycle in $\mathbf{C}^2$, we found the known cases of a parallel M5-brane and the M2-brane which gives the mass of the BPS states of the four-dimensional $\mathcal{N} = 2$ field theory. In particular, BPS monopoles and vortices where found. In addition, there was the case of the "hidden" M5-brane, wrapped on the remaining complex structure of the hyper-Kähler manifold and extended along 89(10). This was related to the structure groups of reduced supersymmetry of M-theory vacua. Since this M5-brane wraps a calibrated cycle in the manifold $\tilde{M}_6$, it has a similar worldvolume theory interpretation as BPS states, with an extra volume modulus. Due to the possible boundary conditions when reduced to Type IIA Hanany-Witten configurations, these BPS vortices and monopoles may have finite tension.

In the case of the background sourced by M5-branes wrapping a holomorphic 2-cycle in $\mathbf{C}^3$, we found the interesting possibility of M5-branes wrapping Cayley calibrated 4-cycles, which changed into M5-branes wrapping associative 3-cycles. This was interpreted as a system of intersecting BPS domain walls. A constraint on the angle of intersection and the angle in charge space was derived. Also, the tension of the domain walls was found to be the integral of a calibrating form in the geometry. A discussion on null structure groups of M-theory vacua with flux showed these arguments to be consistent, providing a physical realisation in terms of M5-branes.

These examples provide a physical realisation of structure groups of M-theory vacua with flux, providing a more intuitive picture in terms of the geometry. It would be interesting to construct more examples of this kind.

Chapter 6

Conclusion

The purpose of this thesis has been to examine various geometric and field theoretic properties of supersymmetric brane probes in supergravity backgrounds consisting of M5-branes wrapped on 2-cycles in $\mathbf{C}^2$ and $\mathbf{C}^3$. In addition, we showed an alternative method of finding supergravity solutions of supersymmetric backgrounds, re-deriving the example of an M5-brane wrapped on a 2-cycle in $\mathbf{C}^3$.

In the first couple of chapters we introduced the necessary concepts needed for a better understanding of the results that followed. We summarised the main features of the supergravity solutions which we examined in the remainder of the thesis. We also provided a brief introduction to G-structures and in particular the bilinear spinor formalism which we used throughout this work. Then followed an introduction to calibrations and their geometric significance. Furthermore, the M5-brane PST action was outlined. The connection to Hanany-Witten models in Type IIA ten-dimensional supergravity was made and a short discussion of these models ensued. The connection to four-dimensional supersymmetric gauge theories was clarified, and a very brief summary of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric Yang-Mills theories was presented.

In Chapter 3 we considered various supersymmetric brane probes of our backgrounds. In particular, we started with an M5-brane probe calculation to determine the metric of the complex scalar kinetic terms of the worldvolume $\mathcal{N} = 2$ effective gauge theory. Using the calibration satisfied by the background, we showed that the metric was indeed Kähler up to boundary terms. We then performed an M2-brane probe calculation to determine the mass of BPS states of the theory. We did this in two ways; firstly from a worldvolume point of view, and secondly from a spinorial derivation. Both these approaches were consistent and determined that there were no supergravity corrections to previous flat space field theory analysis.

In the second half of Chapter 3 we returned to supersymmetric M5-brane probes of our backgrounds. We first looked at the M5-brane probe in the $\mathcal{N} = 2$ background, which corresponded to a D4-brane probe in the appropriate Hanany-Witten model. We calculated the Yang-Mills gauge coupling and theta angle of the gauge theory. Using Euclidean D0-brane probes we also found the instanton Yang-Mills action. We discussed the form of the results and found that at the classical level they agreed with previous flat-space analysis.

We performed the analogous $\mathcal{N} = 1$ calculations for our background of a wrapped M5-brane in $\mathbf{C}^3$. The Yang-Mills gauge coupling was determined, as well as the theta angle and instanton action. We also calculated the metric for the complex scalar kinetic term and using the calibration bound satisfied by this particular background found it was indeed Kähler up to boundary terms. The results were similar to the previous case in that no corrections to the field theory parameters occurred. We also demonstrated an alternative method of obtaining the supergravity solution of our wrapped M5-brane background by the use of bilinear spinors and the differential equations they satisfy. This is related to the G-structures approach of deriving this supergravity solution.

In Chapter 4 we introduced further concepts which were needed for the understanding of the results in the following chapter. Firstly, we extended our notion of calibrations to included calibrating forms of supersymmetric backgrounds with flux and gave an example of a probe M2-brane in a flat background. We also commented on the straightforward construction of these calibrating forms from spinor bilinears. There followed a short discussion of the topological charges which are induced on probe branes in general curved spacetimes. We represented the form of these central charges for both the M2-brane and the M5-brane, explaining their origin.

There followed a discussion of M-theory structure groups, with or without background fluxes. These proved to be important in the geometrical description of brane configurations that were treated in the next chapter. As part of this discussion, we briefly mentioned the refined G-structures approach in the classification of null structure groups. The last part of this chapter comprised a summary of field theory results so that the geometrical results of the next chapter could be interpreted more clearly. We introduced topological gauge theory configurations such as instantons, monopoles, vortices and domain walls, and also gave their brane construction in terms of Type IIA Hanany-Witten models.

In Chapter 5 we calculated the central charges on the worldvolume of M2-branes and M5-branes which were probing our backgrounds of M5-branes wrapped on 2-cycles in $\mathbf{C}^2$ and $\mathbf{C}^3$. In the case of the background sourced by an M5-brane wrapped on a 2-cycle in $\mathbf{C}^2$ we found the known cases of a parallel M5-brane as well as the M2-brane which gives the mass of BPS monopoles and vortices. There was also the case of the "hidden" M5-brane which was analogous to the M2-brane but with a rotation of the complex structure and an extra volume modulus. These possible probes were related to the structure groups of reduced supersymmetry of M-theory vacua. In particular, there seemed to be another manifold with an $SU(3)$ structure that supported this M5-brane probe. The original $SU(5)$ structure seemed to have been reduced to $SU(2) \times SU(3)$ by the action of the branes. There were also terms which belonged to what would be a D6-brane when dimensionally reduced to Type IIA supergravity.

In the case of the background sourced by an M5-brane wrapped on a 2-cycle in $\mathbf{C}^3$ we found the interesting case of M5-branes wrapping Cayley calibrated 4-cycles, which was interpreted as a system of intersecting domain walls in the corresponding $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. A constraint on the angle of intersection and the angle in charge space was derived. The reduced structure group associated with this configuration was $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$ according to the classification. These M5-branes degenerated to M5-branes wrapping associative 3-cycles which realised individual domain walls and which had an associated $(SU(3) \ltimes \mathbb{R}^6) \times \mathbb{R}^3$ group structure. The tension of these domain walls was found to be the integral of a calibrating form of the geometry. A discussion on null structure groups showed these arguments to be consistent, providing a physical realisation in terms of M5-

branes.

As for the future, we are very interested in further applications of the techniques and knowledge learnt from making this thesis. One interesting avenue of research is the question finding solutions of black objects of various topologies (extending the black ring examples [110]) in various lower dimensional supergravities, exploiting the refined G-structures formalism in, for example, seven-dimensional gauged supergravity [111]. This is interesting to attempt since the entropy [112] and microstates [113] of such black objects were found to have a string theory interpretation, as well as a dual CFT description. The M-theory configuration of such black objects could yield further insight into these matters.

It would be very interesting to construct further solutions corresponding to different brane configurations, such as those corresponding to black objects in lower dimensions, using the refined G-structures approach outlined in Chapter 4. The construction of more complicated brane configurations should be possible in practise from knowledge of the Killing spinors preserved by the background. These null structure groups are interesting too because they admit non-static brane solutions such as supertubes [114, 115], supercurves [116, 117] and giant gravitons [118].

Appendix A

Conventions

Throughout this thesis, the eleven dimensions run over $0, 1, 2, \ldots, 8, 9, (10)$ and by convention we take the eleventh dimension to be x^7. We write (10) with brackets to avoid confusion with $1, 0$. So in ten dimensions, the indices would be given by the slightly unnatural 012345689(10). We will use Greek letters $\alpha, \beta, \ldots$ mostly for spinor indices.

In our conventions the epsilon tensor is defined such that

$$\epsilon_{\mu_1 \cdots \mu_n} = 1 = g\epsilon^{\mu_1 \cdots \mu_n}. \tag{A.1}$$

For a d-dimensional manifold, we will generally express p-forms in terms of either the co-ordinate basis $\{dx^0, \ldots, dx^{d-1}\}$, or the orthonormal basis $\{e^0, \ldots, e^{d-1}\}$.

The wedge product of a p-form, w, with a q-form, v, is defined in components by

$$(w \wedge v)_{M_1 \ldots M_{p+q}} = \frac{(p+q)!}{p!q!} w_{[M_1 \ldots M_p} v_{M_{p+1} \ldots M_{p+q}]} \tag{A.2}$$

We also consider the interior product of forms. The definition of the interior product of a q-form, v, and a p-form, w, where necessarily $q > p$, is denoted by $\iota_w v$, and is given in components by

$$(\iota_w v)_{N_1 \ldots N_{q-p}} = \frac{1}{p!} w^{M_1 \ldots M_p} v_{M_1 \ldots M_p N_1 \ldots N_{q-p}} \tag{A.3}$$

The dot product of two p-forms w and v is denoted by $w \cdot w = w^2$ and in components given by

$$w \cdot v = \frac{1}{p!} w_{M_1 \ldots M_p} v^{M_1 \ldots M_p} \tag{A.4}$$

The Lie derivative of a p-form w along the direction specified by the vector X is defined by

$$\mathcal{L}_X w = d(\iota_X w) + \iota_X dw \tag{A.5}$$

We define the complex co-ordinates $z^m = x^m + iy^m$ and also define

$$\epsilon_{M_1 \ldots M_n} \epsilon_{\bar{N}_1 \ldots \bar{N}_n} dz^{M_1} \wedge \ldots dz^{M_n} \wedge dz^{\bar{N}_1} \ldots \wedge dz^{\bar{N}_n} = \epsilon_{\mu_1 \ldots \mu_n} dx^1 \wedge dy^1 \wedge \ldots \wedge dx^n \wedge dy^n. \tag{A.6}$$

A differential form with holomorphic and anti-holomorphic indices is defined as

$$F = \frac{1}{p!q!} F_{M_1 \ldots M_p \bar{N}_1 \ldots \bar{N}_q} dz^{M_1} \wedge \ldots dz^{M_p} \wedge dz^{\bar{N}_1} \wedge \ldots dz^{\bar{N}_q}. \tag{A.7}$$

The Hodge dual on a manifold of the form $\mathbf{R}^{(n)} \times \mathbf{Q}^{(n_C)}$, with $\mathbf{R}^{(n)}$ being an n-dimensional Lorentzian manifold and $\mathbf{Q}^{(n_C)}$ being a hermitian manifold of complex dimension n_C, is defined, for an (r, p, q)-form, as

$$\begin{aligned} *F \;=\; & \frac{\sqrt{g_R}\sqrt{G_C}}{r! m_p! n_q! (n-r)! (n_C - m_p)! (n_C - n_q)!} \epsilon^{\mu_1 \ldots \mu_r}{}_{\mu_{r+1} \ldots \mu_n} \epsilon^{m_1 \ldots m_p}{}_{\bar{m}_{p+1} \ldots \bar{m}_{n_C}} \epsilon^{\bar{n}_1 \ldots \bar{n}_q}{}_{n_{q+1} \ldots n_{n_C}} \\ & dx^{\nu_{r+1}} \wedge \ldots \wedge dx^{\nu_n} \wedge dz^{n_{q+1}} \wedge \ldots \wedge dz^{n_C} \wedge dz^{\bar{m}_{p+1}} \wedge \ldots \wedge dz^{\bar{m}_{n_C}}. \end{aligned} \tag{A.8}$$

The determinant of a hermitian metric can be written in the form

$$\sqrt{\det g_{M\bar{N}}} = \left(\frac{1}{n_C!}\right)^2 \epsilon^{M_1 \ldots M_{n_C}} \epsilon^{\bar{N}_1 \ldots \bar{M}_{n_C}} g_{M_1 \bar{N}_1} \cdots g_{M_{n_C} \bar{N}_{n_C}} \tag{A.9}$$

$$= \left(\frac{1}{n_C!}\right)^2 \left| \epsilon^{M_1 \ldots M_{n_C}} e^1_{M_1} \ldots e^{n_C}_{M_{n_C}} \right|^2. \tag{A.10}$$

This implies that, for example, we have

$$e^1_{[1} e^2_2 e^3_{3]} = 3! \left(\det g_{M\bar{N}}\right)^{1/4}. \tag{A.11}$$

The component of the inverse metric compatible with this definition is

$$g^{M_1\bar{N}_1} = \frac{1}{(n_C-1)!}\frac{1}{\sqrt{\det g_{M\bar{N}}}}\epsilon^{M_1\ldots M_{n_C}}\epsilon^{\bar{N}_1\ldots\bar{N}_{n_C}}g_{M_2\bar{N}_2}\cdots g_{M_{n_C}\bar{N}_{n_C}}. \tag{A.12}$$

A useful identity to know for the calculations of Section 3.4.1 is

$$\partial_y\left(f^{-4}g^{A\bar{B}}\right) = -\partial_y\left(f^4 g_{M\bar{N}}\right)f^{-8}g^{M\bar{B}}g^{A\bar{N}}. \tag{A.13}$$

Bibliography

[1] J. Sanchez Loureda and D. J. Smith, "Four-dimensional SYM probes in wrapped M5-brane backgrounds," hep-th/0509154.

[2] J. S. Loureda and D. J. Smith, "Central charges of wrapped M5-brane backgrounds," hep-th/0604144.

[3] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge, UK: University Press, 2001.

[4] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond.* Cambridge, UK: University Press, 1998.

[5] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory. Vol. 1: Introduction.* Cambridge, UK: University Press, 1987.

[6] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory. Vol. 2: Loop amplitudes, anomalies and phenomenology.* Cambridge, UK: University Press, 1987.

[7] C. V. Johnson, *D-Branes.* Cambridge, UK: University Press, 2003.

[8] J. Polchinski, "Dirichlet-branes and Ramond-Ramond charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, hep-th/9510017.

[9] E. Cremmer, B. Julia, and J. Scherk, "Supergravity theory in 11 dimensions," *Phys. Lett.* **B76** (1978) 409–412.

[10] E. Bergshoeff, E. Sezgin, and P. K. Townsend, "Supermembranes and eleven-dimensional supergravity," *Phys. Lett.* **B189** (1987) 75–78.

[11] R. Gueven, "Black p-brane solutions of D = 11 supergravity theory," *Phys. Lett.* **B276** (1992) 49–55.

[12] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, `hep-th/9711200`.

[13] G. 't Hooft, "Dimensional reduction in quantum gravity," `gr-qc/9310026`.

[14] E. Witten, "Branes and the dynamics of QCD," *Nucl. Phys.* **B507** (1997) 658–690, `hep-th/9706109`.

[15] J. P. Gauntlett, "Classifying supergravity solutions," *Fortsch. Phys.* **53** (2005) 468–479, `hep-th/0501229`.

[16] I. A. Bandos, N. Berkovits, and D. P. Sorokin, "Duality-symmetric eleven-dimensional supergravity and its coupling to M-branes," *Nucl. Phys.* **B522** (1998) 214–233, `hep-th/9711055`.

[17] E. B. Bogomoln'yi, "The Stability of Classical Solutions," *Sov. J. Nucl. Phys.* **24** (1976) 49.

[18] E. Bergshoeff, M. B. Green, G. Papadopoulos, and P. K. Townsend, "The IIA super-eightbrane," `hep-th/9511079`.

[19] E. Bergshoeff, M. de Roo, M. B. Green, G. Papadopoulos, and P. K. Townsend, "Duality of Type II 7-branes and 8-branes," *Nucl. Phys.* **B470** (1996) 113–135, `hep-th/9601150`.

[20] N. Seiberg and E. Witten, "String theory and noncommutative geometry," *JHEP* **09** (1999) 032, `hep-th/9908142`.

[21] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, `hep-th/9905111`.

[22] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Phys. Lett.* **B428** (1998) 105–114, `hep-th/9802109`.

[23] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, `hep-th/9802150`.

[24] J. P. Gauntlett and S. Pakis, "The geometry of D = 11 killing spinors," *JHEP* **04** (2003) 039, `hep-th/0212008`.

[25] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis, and H. S. Reall, "All supersymmetric solutions of minimal supergravity in five dimensions," *Class. Quant. Grav.* **20** (2003) 4587–4634, `hep-th/0209114`.

[26] M. J. Duff and J. T. Liu, "Hidden spacetime symmetries and generalized holonomy in M- theory," *Nucl. Phys.* **B674** (2003) 217–230, `hep-th/0303140`.

[27] J. P. Gauntlett, D. Martelli, and D. Waldram, "Superstrings with intrinsic torsion," *Phys. Rev.* **D69** (2004) 086002, `hep-th/0302158`.

[28] C. Hull, "Holonomy and symmetry in M-theory," `hep-th/0305039`.

[29] J. P. Gauntlett, J. B. Gutowski, and S. Pakis, "The geometry of D = 11 null Killing spinors," *JHEP* **12** (2003) 049, `hep-th/0311112`.

[30] D. Martelli and J. Sparks, "G-structures, fluxes and calibrations in M-theory," *Phys. Rev.* **D68** (2003) 085014, `hep-th/0306225`.

[31] B. Brinne, A. Fayyazuddin, T. Z. Husain, and D. J. Smith, "N = 1 M5-brane geometries," *JHEP* **03** (2001) 052, `hep-th/0012194`.

[32] D. J. Smith, "Intersecting brane solutions in string and M-theory," *Class. Quant. Grav.* **20** (2003) R233, `hep-th/0210157`.

[33] M. Mihailescu, I. Y. Park, and T. A. Tran, "D-branes as solitons of an N = 1, D = 10 non-commutative gauge theory," *Phys. Rev.* **D64** (2001) 046006, `hep-th/0011079`.

[34] E. Witten, "BPS bound states of D0-D6 and D0-D8 systems in a B-field," *JHEP* **04** (2002) 012, `hep-th/0012054`.

[35] E. Witten, "Solutions of four-dimensional field theories via M-theory," *Nucl. Phys.* **B500** (1997) 3–42, `hep-th/9703166`.

[36] R. Harvey and J. Lawson, H. B., "Calibrated geometries," *Acta Math.* **148** (1982) 47.

[37] J. Gutowski and G. Papadopoulos, "AdS calibrations," *Phys. Lett.* **B462** (1999) 81–88, `hep-th/9902034`.

[38] J. Gutowski, G. Papadopoulos, and P. K. Townsend, "Supersymmetry and generalized calibrations," *Phys. Rev.* **D60** (1999) 106006, `hep-th/9905156`.

[39] J. Gutowski, S. Ivanov, and G. Papadopoulos, "Deformations of generalized calibrations and compact non- Kahler manifolds with vanishing first Chern class," `math.dg/0205012`.

[40] G. W. Gibbons and G. Papadopoulos, "Calibrations and intersecting branes," *Commun. Math. Phys.* **202** (1999) 593–619, `hep-th/9803163`.

[41] J. P. Gauntlett, N. D. Lambert, and P. C. West, "Branes and calibrated geometries," *Commun. Math. Phys.* **202** (1999) 571–592, `hep-th/9803216`.

[42] J. P. Gauntlett, N. Kim, and D. Waldram, "M-fivebranes wrapped on supersymmetric cycles," *Phys. Rev.* **D63** (2001) 126001, `hep-th/0012195`.

[43] M. Berger, "Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes," *Bull. Soc. Math France* **83** (1955) 225.

[44] O. Barwald, N. D. Lambert, and P. C. West, "A calibration bound for the M-theory fivebrane," *Phys. Lett.* **B463** (1999) 33–40, `hep-th/9907170`.

[45] M. Cariglia and O. A. P. Mac Conamhna, "Null structure groups in eleven dimensions," `hep-th/0411079`.

[46] A. Fayyazuddin and D. J. Smith, "Localized intersections of M5-branes and four-dimensional superconformal field theories," *JHEP* **04** (1999) 030, `hep-th/9902210`.

[47] A. Fayyazuddin and D. J. Smith, "Warped AdS near-horizon geometry of completely localized intersections of M5-branes," *JHEP* **10** (2000) 023, hep-th/0006060.

[48] B. Brinne, A. Fayyazuddin, S. Mukhopadhyay, and D. J. Smith, "Supergravity M5-branes wrapped on Riemann surfaces and their QFT duals," *JHEP* **12** (2000) 013, hep-th/0009047.

[49] A. Hanany and E. Witten, "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," *Nucl. Phys.* **B492** (1997) 152–190, hep-th/9611230.

[50] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," *Nucl. Phys.* **B426** (1994) 19–52, hep-th/9407087.

[51] A. Fayyazuddin and M. Spalinski, "The Seiberg-Witten differential from M-theory," *Nucl. Phys.* **B508** (1997) 219–228, hep-th/9706087.

[52] S. Elitzur, A. Giveon, and D. Kutasov, "Branes and N = 1 duality in string theory," *Phys. Lett.* **B400** (1997) 269–274, hep-th/9702014.

[53] S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici, and A. Schwimmer, "Brane dynamics and N = 1 supersymmetric gauge theory," *Nucl. Phys.* **B505** (1997) 202–250, hep-th/9704104.

[54] A. A. Tseytlin, "*No-force* condition and BPS combinations of p-branes in 11 and 10 dimensions," *Nucl. Phys.* **B487** (1997) 141–154, hep-th/9609212.

[55] A. Fayyazuddin, T. Z. Husain, and I. Pappa, "The geometry of wrapped M5-branes in Calabi-Yau 2-folds," hep-th/0509018.

[56] K. Hori, H. Ooguri, and Y. Oz, "Strong coupling dynamics of four-dimensional N = 1 gauge theories from M theory fivebrane," *Adv. Theor. Math. Phys.* **1** (1998) 1–52, hep-th/9706082.

[57] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, "Supersymmetric backgrounds from generalized Calabi-Yau manifolds," *JHEP* **08** (2004) 046, `hep-th/0406137`.

[58] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, "Generalized structures of N = 1 vacua," *JHEP* **11** (2005) 020, `hep-th/0505212`.

[59] N. Hitchin, "Generalized Calabi-Yau manifolds," *Quart. J. Math. Oxford Ser.* **54** (2003) 281–308, `math.dg/0209099`.

[60] M. Gualtieri, "Generalized complex geometry," `math.dg/0401221`.

[61] A. Giveon and D. Kutasov, "Brane dynamics and gauge theory," *Rev. Mod. Phys.* **71** (1999) 983–1084, `hep-th/9802067`.

[62] P. Di Vecchia, A. Lerda, and P. Merlatti, "N = 1 and N = 2 super Yang-Mills theories from wrapped branes," *Nucl. Phys.* **B646** (2002) 43–68, `hep-th/0205204`.

[63] C. Nunez, A. Paredes, and A. V. Ramallo, "Flavoring the gravity dual of N = 1 Yang-Mills with probes," *JHEP* **12** (2003) 024, `hep-th/0311201`.

[64] A. Fayyazuddin and T. Z. Husain, "Calibrations, torsion classes and wrapped M-branes," `hep-th/0512030`.

[65] P. Pasti, D. P. Sorokin, and M. Tonin, "Covariant action for a D = 11 five-brane with the chiral field," *Phys. Lett.* **B398** (1997) 41–46, `hep-th/9701037`.

[66] M. Aganagic, J. Park, C. Popescu, and J. H. Schwarz, "World-volume action of the M-theory five-brane," *Nucl. Phys.* **B496** (1997) 191–214, `hep-th/9701166`.

[67] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," *Nucl. Phys.* **B431** (1994) 484–550, `hep-th/9408099`.

[68] J. Wess and J. Bagger, *Supersymmetry and Sueprgravity.* Princeton University Press, 1983.

[69] P. West, *Introduction to Supersymmetry and Supergravity.* Singapore: World Scientific, 1983.

[70] I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity: Or a Walk Through Superspace.* Bristol, UK: IOP, 1998.

[71] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, "Superspace, or one thousand and one lessons in supersymmetry," *Front. Phys.* **58** (1983) 1–548, `hep-th/0108200`.

[72] E. D'Hoker and D. H. Phong, "Lectures on supersymmetric Yang-Mills theory and integrable systems," `hep-th/9912271`.

[73] J. de Boer, K. Hori, H. Ooguri, and Y. Oz, "Kaehler potential and higher derivative terms from M theory five-brane," *Nucl. Phys.* **B518** (1998) 173–211, `hep-th/9711143`.

[74] M. Henningson and P. Yi, "Four-dimensional BPS-spectra via M-theory," *Phys. Rev.* **D57** (1998) 1291–1298, `hep-th/9707251`.

[75] A. Mikhailov, "BPS states and minimal surfaces," *Nucl. Phys.* **B533** (1998) 243–274, `hep-th/9708068`.

[76] B. S. Acharya, J. M. Figueroa-O'Farrill, and B. Spence, "Branes at angles and calibrated geometry," *JHEP* **04** (1998) 012, `hep-th/9803260`.

[77] E. J. Hackett-Jones, D. C. Page, and D. J. Smith, "Topological charges for branes in M-theory," *JHEP* **10** (2003) 005, `hep-th/0306267`.

[78] K. Becker, M. Becker, and A. Strominger, "Five-branes, membranes and nonperturbative string theory," *Nucl. Phys.* **B456** (1995) 130–152, `hep-th/9507158`.

[79] M. Aganagic, J. Park, C. Popescu, and J. H. Schwarz, "Dual D-brane actions," *Nucl. Phys.* **B496** (1997) 215–230, `hep-th/9702133`.

[80] J. L. F. Barbon and A. Pasquinucci, "D0-branes, constrained instantons and D = 4 super Yang- Mills theories," *Nucl. Phys.* **B517** (1998) 125–141, hep-th/9708041.

[81] J. H. Brodie, "Fractional branes, confinement, and dynamically generated superpotentials," *Nucl. Phys.* **B532** (1998) 137–152, hep-th/9803140.

[82] A. Gomberoff, D. Kastor, D. Marolf, and J. H. Traschen, "Fully localized brane intersections: The plot thickens," *Phys. Rev.* **D61** (2000) 024012, hep-th/9905094.

[83] H. Cho, M. Emam, D. Kastor, and J. H. Traschen, "Calibrations and Fayyazuddin-Smith spacetimes," *Phys. Rev.* **D63** (2001) 064003, hep-th/0009062.

[84] T. Z. Husain, "If I only had a brane!," hep-th/0304143.

[85] L. R. Bryant, "Pseudo-Riemannian metrics with parallel spinor fields and non-vanishing Ricci tensor," *Sem. Congr. Soc. Math. France* **4** (2000) 53–94, math.DG/0004073.

[86] J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo, and P. K. Townsend, "Topological extensions of the supersymmetry algebra for extended objects," *Phys. Rev. Lett.* **63** (1989) 2443.

[87] P. K. Townsend, "M-theory from its superalgebra," hep-th/9712004.

[88] T. Sato, "Various supersymmetric brane configurations from superalgebras in many types of M-brane backgrounds," *Nucl. Phys.* **B548** (1999) 231–257, hep-th/9812014.

[89] J. Gillard, U. Gran, and G. Papadopoulos, "The spinorial geometry of supersymmetric backgrounds," *Class. Quant. Grav.* **22** (2005) 1033–1076, hep-th/0410155.

[90] J. M. Figueroa-O'Farrill, "Breaking the M-waves," *Class. Quant. Grav.* **17** (2000) 2925–2948, hep-th/9904124.

[91] O. A. P. Mac Conamhna, "Refining G-structure classifications," *Phys. Rev.* **D70** (2004) 105024, hep-th/0408203.

[92] D. Tong, "TASI lectures on solitons," hep-th/0509216.

[93] A. A. Belavin, A. M. Polyakov, A. S. Shvarts, and Y. S. Tyupkin, "Pseudoparticle solutions of the Yang-Mills equations," *Phys. Lett.* **B59** (1975) 85–87.

[94] M. A. Shifman and A. I. Vainshtein, "Instantons versus supersymmetry: Fifteen years later," hep-th/9902018.

[95] N. Dorey, T. J. Hollowood, V. V. Khoze, and M. P. Mattis, "The calculus of many instantons," *Phys. Rept.* **371** (2002) 231–459, hep-th/0206063.

[96] E. Witten, "Dyons of charge e theta / 2 pi," *Phys. Lett.* **B86** (1979) 283–287.

[97] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Y. I. Manin, "Construction of instantons," *Phys. Lett.* **A65** (1978) 185–187.

[98] J. A. Harvey, "Magnetic monopoles, duality, and supersymmetry," hep-th/9603086.

[99] T. T. Wu and C. N. Yang, "Dirac monopole without strings: Monopole harmonics," *Nucl. Phys.* **B107** (1976) 365.

[100] A. Hanany and D. Tong, "Vortices, instantons and branes," *JHEP* **07** (2003) 037, hep-th/0306150.

[101] E. R. C. Abraham and P. K. Townsend, "Q kinks," *Phys. Lett.* **B291** (1992) 85–88.

[102] E. R. C. Abraham and P. K. Townsend, "More on Q kinks: A (1+1)-dimensional analog of dyons," *Phys. Lett.* **B295** (1992) 225–232.

[103] M. Shifman and A. Yung, "Domain walls and flux tubes in N = 2 SQCD: D-brane prototypes," *Phys. Rev.* **D67** (2003) 125007, hep-th/0212293.

[104] D. Tong, "The moduli space of BPS domain walls," *Phys. Rev.* **D66** (2002) 025013, hep-th/0202012.

[105] N. Sakai and Y. Yang, "Moduli sapce of BPS walls in supersymmetric gauge theories," hep-th/0505136.

[106] N. D. Lambert and D. Tong, "Kinky D-strings," *Nucl. Phys.* **B569** (2000) 606–624, hep-th/9907098.

[107] M. Eto *et al.*, "D-brane construction for non-Abelian walls," *Phys. Rev.* **D71** (2005) 125006, hep-th/0412024.

[108] A. Hanany and D. Tong, "On monopoles and domain walls," hep-th/0507140.

[109] J. P. Gauntlett, G. W. Gibbons, and P. K. Townsend, "Intersecting domain walls in MQCD," *Phys. Lett.* **B483** (2000) 240–247, hep-th/0004136.

[110] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall, "A supersymmetric black ring," *Phys. Rev. Lett.* **93** (2004) 211302, hep-th/0407065.

[111] M. Cariglia and O. A. P. Mac Conamhna, "Timelike Killing spinors in seven dimensions," *Phys. Rev.* **D70** (2004) 125009, hep-th/0407127.

[112] R. Emparan, "Black hole entropy as entanglement entropy: A holographic derivation," hep-th/0603081.

[113] S. D. Mathur, "The fuzzball proposal for black holes: An elementary review," *Fortsch. Phys.* **53** (2005) 793–827, hep-th/0502050.

[114] D. Mateos and P. K. Townsend, "Supertubes," *Phys. Rev. Lett.* **87** (2001) 011602, hep-th/0103030.

[115] R. Emparan, D. Mateos, and P. K. Townsend, "Supergravity supertubes," *JHEP* **07** (2001) 011, hep-th/0106012.

[116] D. Mateos, S. Ng, and P. K. Townsend, "Supercurves," *Phys. Lett.* **B538** (2002) 366–374, hep-th/0204062.

[117] Y. Hyakutake and N. Ohta, "Supertubes and supercurves from M-ribbons," *Phys. Lett.* **B539** (2002) 153–161, `hep-th/0204161`.

[118] A. Mikhailov, "Giant gravitons from holomorphic surfaces," *JHEP* **11** (2000) 027, `hep-th/0010206`.